D-instantons in Calabi-Yau compactifications

Sergei Alexandrov

Laboratoire Charles Coulomb, CNRS, Montpellier

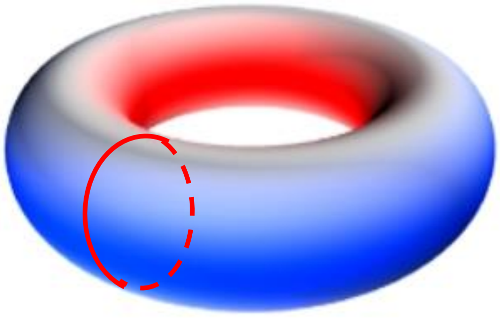
S.A., A.Sen, B.Stefanski arXiv:2108.04265
arXiv:2110.06949

See talk of Manki Kim for the follow-up on D-instantons in CY orientifolds

S.A., A.Firat, M.Kim, A.Sen, B.Stefanski arXiv:2204.02981

String Field Theory 2022
Prague

Motivation



Instantons in string theory – *Euclidean branes* wrapped on non-trivial cycles of compactification manifold

Although exponentially suppressed in small g_s limit, they play important role for various reasons:

- crucial for non-perturbative dualities and for going beyond the perturbative formulation
- essential for moduli stabilization
- contain information on numerical invariants of compactification manifold

entropy of BPS black holes, modularity and all that…

But in contrast to gauge theories, until 2020, *a direct computation* of instanton effects suffered from *ambiguities*!!!

Breakthrough: understanding infrared and zero mode divergences through string field theory [A.Sen]

→ *A recipe how to deal with zero modes to obtain finite results*

Goal

These ideas have been applied to:

Perfect *match* with results based on

known before

- non-critical string theory
  [Sen `21, Eniceicu,Mahajan,Murdia,Sen `22, Chakravarty,Sen `22]

  matrix models

- Type IIB string theory in flat space
  [Sen `21]

  S-duality

- Type II string theory on a Calabi-Yau threefold
  [S.A.,Sen,Stefanski `21]

  Supersymmetry, S-duality, mirror symmetry

- Type II string theory on a Calabi-Yau orientifold
  [S.A.,Firat,Kim,Sen,Stefanski `22]

  see the talk of Manki Kim

  → **new result**

  D-instanton induced superpotential

I'll explain
- what we compute
- what was known
- how we compute

The plan of the talk

1. Review of instanton corrections in CY compactifications
   → Hypermultiplet metric and D-instantons

2. D-instanton corrections to string amplitudes: problems and their resolution

3. Computation of (some) relevant contributions in type IIA

4. D-instantons in Type IIB

5. Conclusions

Instanton corrections in CY compactifications

The effective action of Type II string theory on a CY threefold is determined by the metric on the moduli space $\mathcal{M}_{\rm VM} \times \mathcal{M}_{\rm HM}$

special Kähler
no corrections in string coupling

quaternion-Kähler
corrections in string coupling

- **Classical metric (c-map)**

$$\mathrm{d}s^2 = \mathrm{d}\phi^2 - e^{\phi}\,(\mathrm{Im}\mathcal{N}^{-1})^{\Lambda\Sigma}\left(\mathrm{d}\tilde\zeta_\Lambda - \mathcal{N}_{\Lambda\Lambda'}\mathrm{d}\zeta^{\Lambda'}\right)\left(\mathrm{d}\tilde\zeta_\Sigma - \bar{\mathcal{N}}_{\Sigma\Sigma'}\mathrm{d}\zeta^{\Sigma'}\right) + \tfrac{1}{4}\,e^{2\phi}\left(\mathrm{d}\sigma + \tilde\zeta_\Lambda \mathrm{d}\zeta^\Lambda - \zeta^\Lambda \mathrm{d}\tilde\zeta_\Lambda\right)^2 + 4\mathcal{K}_{a\bar b}\,\mathrm{d}z^a \mathrm{d}\bar z^{\bar b}$$

dilaton $(e^{\phi} \sim g_s^2)$

NS-axion

RR-scalars

complex structure moduli (IIA)
complexified Kähler moduli (IIB)

$\mathcal{N}_{\Lambda\Sigma},\ \mathcal{K}$ – determined by holomorphic prepotential $F(z)$

- **1-loop correction**

[Antoniadis,Minasian,Theisen,Vanhove '03
Robles-Llana,Saueressig,Vandoren '06, S.A. '07]

- **D-brane instantons**

$$e^{-2\pi|Z_\gamma|/g_s - 2\pi \mathrm{i}(q_\Lambda\zeta^\Lambda - p^\Lambda\tilde\zeta_\Lambda)}$$

$d_{\rm cycle}$	0	1	2	3	4	5	6
IIA :		×		D2		×	
IIB :	D(-1)		D1		D3		D5

- **NS5-brane instantons**

$$e^{-2\pi|k|\mathcal{V}/g_s^2 - \mathrm{i}\pi k\sigma}$$

Instanton corrections from dualities

Chain of dualities:

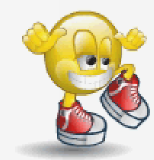
[Robles-Llana, Saueressig, Theis, Vandoren '07] {α', 1-loop }

↓ S-duality

{A-D2} ← mirror — {α', D(-1), D1}

IIA e/m ↓ duality IIB

{all D2} — mirror → {α', D(-1), D1, D3, D5}



↓ S-duality

{D2, NS5} ← mirror — {α', D(-1), D1, D3, D5, NS5}

Where we are:

Type IIA		Type IIB	
D2	to all orders	D1-D(-1)	to all orders
NS5	?	D3, D5-NS5	one-instanton approximation

D-instanton corrected metric

Instanton corrections are encoded into the *holomorphic contact structure* on the *twistor space* over $\mathcal{M}_{\rm HM}$

In practice: they can be extracted from *holomorphic Darboux coordinates* determined as solutions of integral equations

[S.A.,Pioline,Saueressig,Vandoren '08]

$$\mathcal{X}_\gamma(t) = \mathcal{X}^{\rm sf}_\gamma(t) \exp\left[\frac{1}{4\pi \mathrm{i}} \sum_{\gamma'} \Omega_{\gamma'} \langle\gamma,\gamma'\rangle \int_{\ell_\gamma} \frac{\mathrm{d}t'}{t'} \frac{t+t'}{t-t'} \log\left(1-\mathcal{X}_{\gamma'}(t')\right)\right]$$

$$\mathcal{X}^{\rm sf}_\gamma(t) = e^{-\frac{2\pi \mathrm{i}}{g_s}\left(t^{-1}Z_\gamma - t\bar Z_\gamma\right) - 2\pi \mathrm{i}(q_\Lambda \zeta^\Lambda - p^\Lambda \tilde\zeta_\Lambda)}$$

$\gamma = (p^\Lambda, q_\Lambda)$ – D-brane charge

$Z_\gamma = q_\Lambda z^\Lambda - p^\Lambda F_\Lambda$ – central charge

DT invariants of CY

the same as integral equation of Gaiotto-Moore-Neitzke for N=2 SYM / S^1

Given $\mathcal{X}_\gamma(t)$, there is a long and tedious procedure to extract the metric

It was realized in [S.A., Banerjee '14]

Small string coupling limit

We are interested only in the leading corrections in the small g_s limit in a given topological sector, i.e. for fixed axionic coupling $\Theta_\gamma \equiv q_\lambda \zeta^\Lambda - p^\Lambda \tilde{\zeta}_\Lambda$

All terms *non-linear* in DT invariants are subleading (no need to solve integral equations)

$$\mathrm{d}s^2_{\mathrm{inst}} = \sum_\gamma \frac{\Omega_\gamma\, e^{(5\phi-\mathcal{K})/4}}{64\pi\sqrt{|Z_\gamma|}} \left(\sum_{k=1}^{\infty} k^{-1/2}\, e^{-k\mathcal{T}_\gamma}\right)\left(\mathcal{A}^2_\gamma + (\cdots)\mathrm{d}\mathcal{T}_\gamma\right)$$

such terms will be ignored

$\mathcal{T}_\gamma = 8\pi\, e^{(\mathcal{K}-\phi)/2}|Z_\gamma| + 2\pi \mathrm{i}\Theta_\gamma$ – instanton action

$$\mathcal{A}_\gamma = |Z_\gamma| e^{(\mathcal{K}+\phi)/2}\left(\mathrm{d}\sigma + \tilde{\zeta}_\Lambda \mathrm{d}\zeta^\Lambda - \zeta^\Lambda \mathrm{d}\tilde{\zeta}_\Lambda + 8e^{-\phi}\mathrm{Im}\partial\log(e^{\mathcal{K}}Z_\gamma)\right) - 4\mathrm{i}\,\mathcal{C}_\gamma$$

where $$\mathcal{C}_\gamma = -\frac{1}{2}(\mathrm{Im}F)^{\Lambda\Sigma}\left(q_\Lambda - \mathrm{Re}F_{\Lambda\Xi}p^\Xi\right)\left(\mathrm{d}\tilde{\zeta}_\Sigma - \mathrm{Re}F_{\Sigma\Theta}\mathrm{d}\zeta^\Theta\right) - \frac{1}{2}\mathrm{Im}F_{\Lambda\Sigma}\,p^\Lambda\,\mathrm{d}\zeta^\Sigma$$

Instanton corrections to string amplitudes

Sen '20

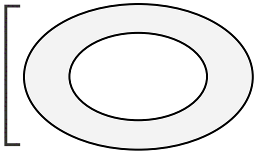
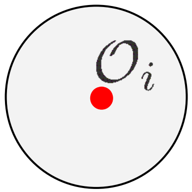
Amplitudes in the presence of instantons:

$$\langle \mathcal{O} \rangle = \frac{\int d\varphi_{\rm p} e^{S_{\rm p}} \mathcal{O} + e^{-\mathcal{T}} \int d\varphi_{\rm i} e^{S_{\rm in}} \mathcal{O}}{\int d\varphi_{\rm p} e^{S_{\rm p}} + e^{-\mathcal{T}} \int d\varphi_{\rm i} e^{S_{\rm in}}} \approx \langle \mathcal{O} \rangle_{\rm p} + e^{-\mathcal{T}} \left[\frac{\int d\varphi_{\rm i} e^{S_{\rm in}} \mathcal{O}}{\int d\varphi_{\rm p} e^{S_{\rm p}}} - \frac{\int d\varphi_{\rm p} e^{S_{\rm p}} \mathcal{O}}{\int d\varphi_{\rm p} e^{S_{\rm p}}} \frac{\int d\varphi_{\rm i} e^{S_{\rm in}}}{\int d\varphi_{\rm p} e^{S_{\rm p}}} \right]$$

includes *disconnected and bubble* diagrams such that

- each have either a vertex operator insertion or a boundary on D-instanton
- at least one diagram has both

The leading instanton contribution to n-point function:

$$\left\langle \prod_{i=1}^{n} \mathcal{O}_i \right\rangle_{\rm inst} = e^{-\mathcal{T}} \exp \Bigg[$$

$$\Bigg] \prod_{i=1}^{n}$$


But there are problems:

- there are divergences due to zero modes
- the annulus amplitude formally vanishes

Example: in type IIB in 10d

$$\int_0^\infty \frac{dt}{2t} \left[\frac{1}{2} \eta(\mathrm{i}t)^{-12} \left(\vartheta_3(0,\mathrm{i}t)^4 - \vartheta_4(0,\mathrm{i}t)^4 - \vartheta_2(0,\mathrm{i}t)^4 + \vartheta_1(0,\mathrm{i}t)^4 \right) \right] = 0$$

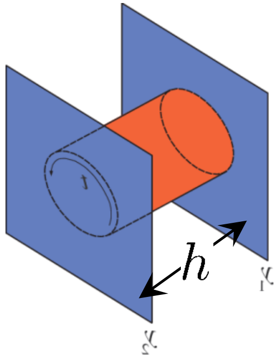
Recipe

All divergences can be understood from *string field theory* [Sen ’20]

- the zero modes related to the collective coordinates of the D-instanton should be left unintegrated till the end of calculation
  - bosonic zero modes produce the momentum conserving delta-function
  - fermionic zero modes require insertion of zero mode vertex operators into the disk diagrams
- the divergence due to ghost zero modes arises due to the braekdown of the Seigel gauge $b_0|\Psi\rangle = 0$ used to get the worldsheet formulation, which is cured by working with a gauge invariant path integral
- other zero modes (if present) are integrated with a non-linear effective action

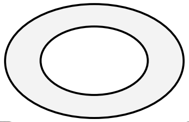
To extract a finite result — regularization: *shifted boundary conditions*

$$L_0|\text{zero mode}\rangle = h|\text{zero mode}\rangle$$



After resolving all issues, the regularization can be safely removed $h \to 0$

Annulus amplitude from string field theory

$\exp\Big[$  $\Big]$

$$= i\Omega_\gamma \int_{-\infty}^{\infty} d\phi\, e^{-\phi^2/4} \int \prod_\mu d\tilde{\xi}^\mu \int \prod_{\delta,\dot{\delta}=1}^{2} d\tilde{\chi}^\delta\, d\tilde{\chi}^{\dot{\delta}} \Big/ \int d\theta$$

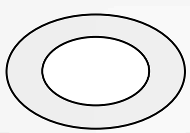
- $i\Omega_\gamma$: counts cycles
- $\int_{-\infty}^{\infty} d\phi\, e^{-\phi^2/4}$: ghost mode due to unfixing Seigel gauge → $2\sqrt{\pi}$
- $\int \prod_\mu d\tilde{\xi}^\mu$: bosonic z.m. → $\tilde{\xi}^\mu = \dfrac{\xi^\mu}{\sqrt{2}\,\pi g_o}$
- $\int \prod_{\delta,\dot{\delta}=1}^{2} d\tilde{\chi}^\delta\, d\tilde{\chi}^{\dot{\delta}}$: fermionic z.m. → $\tilde{\chi}^\alpha = g_o^{-1}\chi^\alpha$
- $\int d\theta$: volume of the unfixed U(1) gauge group → $4\pi/g_o$

ξ^μ — spacetime coordinate of D-instanton

χ^α — normalized f.z.m.

g_o — open string coupling

$$\mathrm{Re}\mathcal{T}_\gamma = \frac{1}{2\pi^2 g_o^2} \;\longrightarrow\; g_o^2 = \frac{2g_s}{V_\gamma} = \frac{e^{(\phi-\mathcal{K})/2}}{16\pi^3 |Z_\gamma|}$$

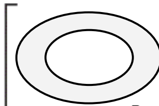
$\exp\Big[$  $\Big]$

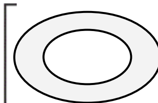
$$= i\, C\, g_o\, \Omega_\gamma \int \prod_\mu d\xi^\mu \int \prod_{\delta,\dot{\delta}=1}^{2} d\chi^\delta\, d\chi^{\dot{\delta}}$$

$$C = 2^{-5}\pi^{-13/2}$$

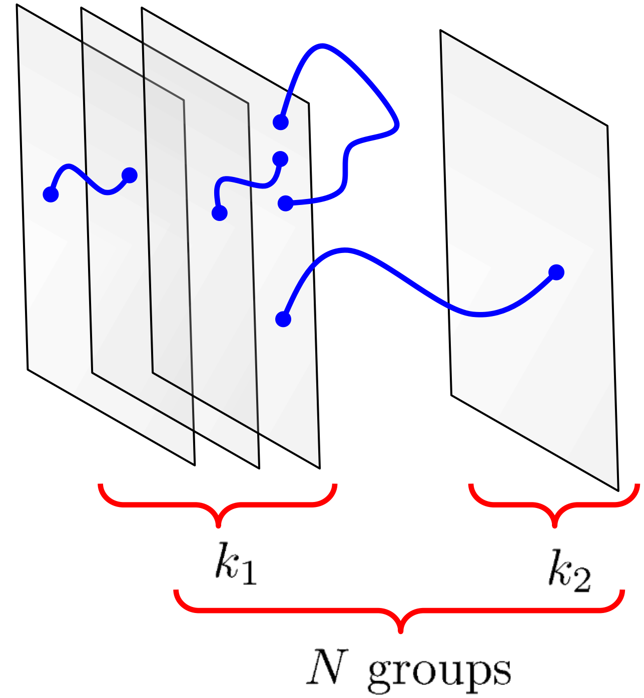
- $\int \prod_\mu d\xi^\mu$ → $(2\pi)^4 \delta^{(4)}\left(\sum_i p_i\right)$
- $\int \prod d\chi^\delta\, d\chi^{\dot{\delta}}$ → saturated by vertex operators inserted into discs diagrams

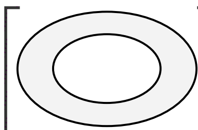
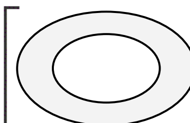
Annulus amplitude for multiple instantons

For coincident branes there are additional zero modes → $\exp\Big[$  $\Big]_{k_i} \sim g_o$

→ total amplitude $\exp\Big[$  $\Big] \sim g_o^N$

→ the leading contribution comes from configuration with $N = 1$ and $\sim g_o$



$\exp\Big[$  $\Big]_k = \mathcal{N}_k \exp\Big[$  $\Big]_1$

$$\mathcal{N}_k = \frac{(\sqrt{\pi})^{k^2-1}}{(2\pi)^{2(k^2-1)}} \frac{k^{3/2}}{V_{SU(k)/\mathbb{Z}_k}} \int \prod_{a=1}^{k^2-1} \left\{ \prod_{\mu=0}^{3} dx_a^\mu \right\} \left\{ \prod_{\delta,\dot{\delta}=1}^{2} dy_a^\delta dy_a^{\dot{\delta}} \right\} e^{S(x_a,y_a)}$$

additional zero modes

effective action 4d N=1 SYM ↘ 0d

$$\boxed{\mathcal{N}_k = k^{-1/2}}$$

← the integral is evaluated in [Krauth,Nicolai,Staudacher '98, Moore,Nekrasov,Shatashvili '98]

Metric vs. curvature

We are interested in the effective action for massless scalars. For such fields, 2- and 3-point amplitudes vanish → we need *4-point* function

The simplest 4-point function affected by the metric on $\mathcal{M}_{\rm HM}$ is generated by

$$\int d^4x\, \mathcal{R}_{ijkl}(\chi^i\bar{\chi}^j)(\chi^k\bar{\chi}^l)$$

Symmetric part of (the *Sp(n)* part of) the curvature on $\mathcal{M}_{\rm HM}$
very complicated!

fermions from hypermultiplets

Solution:
do not impose momentum conservation!

General structure I

The effective action: $$-\frac{1}{2}\int d^4x\, G_{ij}(\vec{\varphi})\,\partial_\mu\varphi^i\partial^\mu\varphi^j$$

$$G_{ij} = g_{ij} + \sum_\gamma e^{-\mathcal{T}_\gamma}\left(h_{ij}^{(\gamma)} + \cdots\right)$$

$$\varphi^i = \phi^i + \lambda^i$$ ← quantum fluctuations

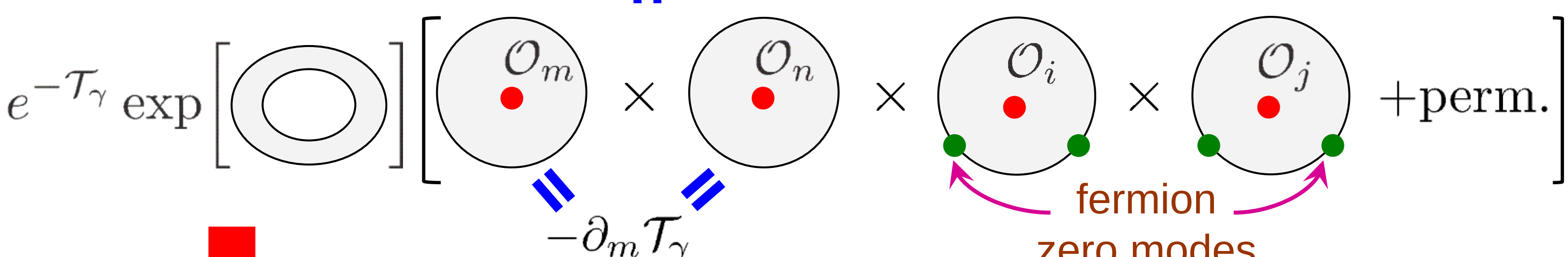
The leading 4-λ term: $$-\frac{1}{4}\int d^4x\, e^{-\mathcal{T}_\gamma}\,\partial_m\mathcal{T}_\gamma\,\partial_n\mathcal{T}_\gamma\, h_{ij}^{(\gamma)}(\vec{\phi})\,\lambda^m\,\lambda^n\,\partial_\mu\lambda^i\partial^\mu\lambda^j$$

because $\mathcal{T}_\gamma \sim 1/g_s$

$$\left\langle \mathcal{O}_m\mathcal{O}_n\mathcal{O}_i\mathcal{O}_j\right\rangle = (2\pi)^4\,\delta^{(4)}\left(\sum_\alpha p^{(\alpha)}\right) e^{-\mathcal{T}_\gamma}\left[\partial_m\mathcal{T}_\gamma\,\partial_n\mathcal{T}_\gamma\, h_{ij}^{(\gamma)}(\vec{\phi})\,(p_i\cdot p_j) + \text{perm.}\right]$$

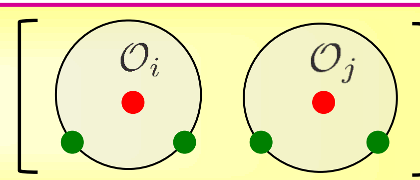
=

from string instanton calculus

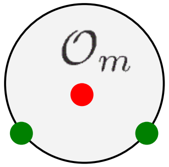


vanishes for $p_i = p_j$

$$(p_i\cdot p_j)\,h_{ij}^{(\gamma)} = \sum_{k|\gamma}\frac{C}{\sqrt{k}}\,g_o\,\Omega_{\gamma/k}\int\prod_{\delta,\dot{\delta}=1}^{2} d\chi_a^\delta d\chi_a^{\dot{\delta}}$$



General structure II



$$= \mathrm{i}\, a_m(\vec{\varphi})\, p_\mu\, \gamma^\mu_{\dot\alpha\alpha}\, \chi^\alpha \chi^{\dot\alpha}$$

$$ds^2_{\rm inst} = \sum_\gamma 2\pi e^\phi\, g_o\, \Omega_\gamma \left(\sum_{k=1}^{\infty} k^{-1/2}\, e^{-k\mathcal{T}_\gamma} \right) \left(\sum_m a_m\, d\lambda^m \right)^2 + \mathcal{O}(d\mathcal{T}_\gamma)$$

Caveat: this procedure is insensitive to the field redefinitions

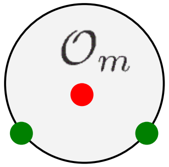
$$\varphi^m \to \varphi^m + e^{-\mathcal{T}_\gamma}\, \xi^m(\vec{\varphi})$$

leading order

$$d\varphi^m \to d\varphi^m - e^{-\mathcal{T}_\gamma}\, \xi^m(\vec{\varphi})\, d\mathcal{T}_\gamma$$

Terms $\sim d\mathcal{T}_\gamma$ cannot be compared

Example 1: NS-axion contribution

It remains to compute 

$$= \mathrm{i}\, a_m(\vec{\varphi})\, p_\mu\, \gamma^\mu_{\dot\alpha\alpha}\, \chi^\alpha \chi^{\dot\alpha}$$

for $m = \sigma,\ (\zeta^\Lambda, \tilde\zeta_\Lambda),\ z^a,\ \phi$

NS-axion is the scalar dual to the *B* field. Thus, we have to analyze:

$$\mathrm{i}\pi\kappa\, T_2\, \mathcal{X}^\alpha \hat{\mathcal{X}}^\beta \int_{-\infty}^{\infty} dz \left\langle V_B(\mathrm{i})\ c e^{-\phi/2} S_\alpha(0)\ e^{-\phi/2} S_\beta(z) \right\rangle$$

grav. coupling.

brane tension.

10d ferm. z.m.

B-field vertex operator

unintegrated and integrated vertex op. of open str. fermions

$$V_B = 2\, b_{\mu\nu}\, c\, \bar c\, (\partial X^\mu + \mathrm{i}\, p_\rho\, \psi^\rho \psi^\mu)\, e^{\mathrm{i}p\cdot X} e^{-\bar\phi} \bar\psi^\nu + \cdots$$

- doubling trick
- OPE
- contour integration
- $\mathcal{X}^\alpha = \eta \otimes \chi^\alpha$ and $\hat{\mathcal{X}}^\alpha = \bar\eta \otimes \chi^{\dot\alpha}$

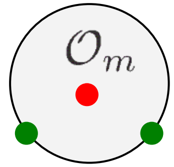
covariantly constant spinor on CY

$$\frac{1}{2}\, \pi^2 \kappa\, T_2\, p_\rho\, b_{\mu\nu}\, \chi^\alpha \chi^{\dot\alpha}\, (\gamma^{\rho\mu\nu})_{\dot\alpha\alpha}\, (2\pi)^3 \delta^{(3)}(0)$$

artefact of the flat space

Example 1: NS-axion contribution

It remains to compute



$$= \mathrm{i}\, a_m(\vec{\varphi})\, p_\mu\, \gamma^\mu_{\dot\alpha\alpha}\, \chi^\alpha \chi^{\dot\alpha}$$

for $m = \sigma,\ (\zeta^\Lambda, \tilde\zeta_\Lambda),\ z^a,\ \phi$

NS-axion is the scalar dual to the *B* field. Thus, we have to analyze:

$$\mathrm{i}\pi\kappa\, T_2\, \mathcal{X}^\alpha \hat{\mathcal{X}}^\beta \int_{-\infty}^{\infty} dz \left\langle V_B(\mathrm{i})\ c e^{-\phi/2} S_\alpha(0)\ e^{-\phi/2} S_\beta(z) \right\rangle$$

- grav. coupling.
- brane tension.
- 10d ferm. z.m.
- *B*-field vertex operator
- unintegrated and integrated vertex op. of open str. fermions

$$V_B = 2\, b_{\mu\nu}\, c\, \bar c\, (\partial X^\mu + \mathrm{i}\, p_\rho\, \psi^\rho \psi^\mu)\, e^{\mathrm{i}p\cdot X} e^{-\bar\phi} \bar\psi^\nu + \cdots$$

- doubling trick
- OPE
- contour integration
- $\mathcal{X}^\alpha = \eta \otimes \chi^\alpha$ and $\hat{\mathcal{X}}^\alpha = \bar\eta \otimes \chi^{\dot\alpha}$

covariantly constant spinor on CY

$$\frac{1}{2}\, \pi^2 \kappa\, T_2\, p_\rho\, b_{\mu\nu}\, \chi^\alpha \chi^{\dot\alpha}\, (\gamma^{\rho\mu\nu})_{\dot\alpha\alpha}\ V_\gamma$$

volume of the wrapped cycle

Dualization

But we still have to go from the *B* field to the NS axion

- generalize the previous construction from scalars to 2-forms
- perform dualization in the presence of instanton corrections

$$\int d^4x \left(\frac{f}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} - \underbrace{\frac{1}{6}\, \varepsilon^{\mu\nu\rho\tau} H_{\mu\nu\rho} A_\tau}_{\text{CS type of coupling}} \right) \quad \overset{\text{duality}}{\Longleftrightarrow} \quad \int \frac{d^4x}{2f} \left(\partial_\mu \sigma + A_\mu\right)\left(\partial^\mu \sigma + A^\mu\right)$$

$$H^{\mu\nu\rho} = f^{-1}\, \varepsilon^{\mu\nu\rho\tau} \left(\partial_\tau \sigma + A_\tau\right)$$

The dual actions are not equal, but their deformations due to $f \to f + \delta f$ are *equal* (to the first order) on the undeformed duality relation

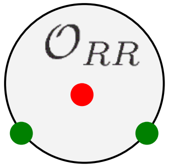
It is enough to substitute into the disk 1-point function

$$\partial_{[\rho} b_{\mu\nu]} = \frac{\kappa}{12\pi V}\, \varepsilon_{\rho\mu\nu}{}^{\tau} \Big(\partial_\tau \sigma + \underbrace{\tilde{\zeta}_\Lambda \partial_\tau \zeta^\Lambda - \zeta^\Lambda \partial_\tau \tilde{\zeta}_\Lambda}_{A_\tau} \Big)$$

$$T_2 V_\gamma = \mathrm{Re}\, \mathcal{T}_\gamma$$

$$a_\sigma \nabla \sigma = \frac{\pi \kappa^2}{4V}\, \mathrm{Re}\, \mathcal{T}_\gamma \left(d\sigma + \tilde{\zeta}_\Lambda\, d\zeta^\Lambda - \zeta^\Lambda\, d\tilde{\zeta}_\Lambda \right)$$

Example 2: RR-field contribution



$$= \mathrm{i}\pi\kappa\, T_2\, \mathcal{X}^{\alpha}\hat{\mathcal{X}}^{\beta} \int_{-\infty}^{\infty} dz \left\langle V_{RR}(\mathrm{i})\ c e^{-\phi/2} S_{\alpha}(0)\ e^{-\phi/2} S_{\beta}(z) \right\rangle$$

New ingredient: for the doubling trick, we need to use the boundary condition for the spin field

$$\bar{S}^{\delta} = \frac{1}{3!}\, v_{\gamma,IJK}\, (\Gamma^{IJK})^{\delta\delta'} S_{\delta'}$$

volume form of the wrapped 3-cycle

$$-\mathrm{i}\,\frac{\pi^2}{36}\,\kappa\, T_2\, \chi^{\alpha}\chi^{\dot\beta}\, p_{\mu} \int_{L_\gamma} d^3x\ \tilde{C}_{\mathbf{pqr}}\, v_{\gamma,\mathbf{ijk}}\, (\bar\eta\Gamma^{M}\Gamma^{\mu\mathbf{pqr}}\Gamma^{\mathbf{ijk}}\Gamma_{M}\eta)_{\dot\beta\alpha}$$

Kähler form

on CY $\quad \bar\eta\Gamma^{\mathbf{ij}}\eta = \mathrm{i}\omega^{\mathbf{ij}} = -\mathrm{i}J^{\mathbf{i}}_{\mathbf{k}}g^{\mathbf{kj}}$

complex structure

$$-4\mathrm{i}\,\pi^2\kappa\, T_2\, (\gamma^{\mu})_{\dot\beta\alpha}\, \chi^{\alpha}\chi^{\dot\beta}\, p_{\mu} \int_{L_\gamma} \left(2\,\tilde{C} + \mathrm{i}\star\tilde{C} - 3\mathrm{i}\, J(\tilde{C})\right)$$

One must fix the relative normalization of $\tilde{C}$ and C giving rise to

$\zeta^{\Lambda},\ \tilde{\zeta}_{\Lambda}$

$$= -2\pi\mathrm{i} \int_{L_\gamma} C \quad \Rightarrow \quad \tilde{C} = \frac{\pi}{4\kappa\, T_2}\, C$$

Evaluation of the integrals

It remains to evaluate 3 integrals:

$$\int_{L_\gamma} C \;= q_\Lambda \zeta^\Lambda - p^\Lambda \tilde\zeta_\Lambda = \Theta_\gamma$$

$$\int_{L_\gamma} \star C \;= 2\mathscr{C}_\gamma - 4e^{\mathcal{K}}\mathrm{Re}\left[\bar Z_\gamma (z^\Lambda \tilde\zeta_\Lambda - F_\Lambda \zeta^\Lambda)\right]$$

$$\int_{L_\gamma} J(C) = -\frac{2}{3}\mathscr{C}_\gamma - \frac{4}{3K}\mathrm{Re}\left[\bar Z_\gamma (z^\Lambda \tilde\zeta_\Lambda - F_\Lambda \zeta^\Lambda)\right]$$

Basis: $\{A^\Lambda, B_\Lambda\} \in H_3(\mathfrak{Y})$

$\{\alpha_\Lambda, \beta^\Lambda\} \in H^3(\mathbb{Z}, \mathfrak{Y})$

$$\int_{A^\Lambda} \alpha_\Sigma = -\int_{B_\Sigma} \beta^\Lambda = \delta^\Lambda_\Sigma, \quad \int_{B_\Lambda} \alpha_\Sigma = \int_{A^\Lambda} \beta^\Sigma = 0$$

$$L_\gamma = q_\Lambda A^\Lambda - p^\Lambda B_\Lambda$$

$$C = \zeta^\Lambda \alpha_\Lambda - \tilde\zeta_\Lambda \beta^\Lambda$$

$\Omega = z^\Lambda \alpha_\Lambda - F_\Lambda \beta^\Lambda$ – hol. form of CY

where $\mathscr{C}_\gamma = -\frac{1}{2}(\mathrm{Im}F)^{\Lambda\Sigma}\left(q_\Lambda - \mathrm{Re}F_{\Lambda\Xi}p^\Xi\right)\left(\tilde\zeta_\Sigma - \mathrm{Re}F_{\Sigma\Theta}\zeta^\Theta\right) - \frac{1}{2}\mathrm{Im}F_{\Lambda\Sigma}\, p^\Lambda \zeta^\Sigma$

$$\star\alpha_\Lambda = \left(\mathrm{Re}\mathcal{N}(\mathrm{Im}\mathcal{N})^{-1}\right)_\Lambda^{\ \Sigma}\alpha_\Sigma - \left(\mathrm{Im}\mathcal{N} + \mathrm{Re}\mathcal{N}(\mathrm{Im}\mathcal{N})^{-1}\mathrm{Re}\mathcal{N}\right)_{\Lambda\Sigma}\beta^\Sigma$$

Suzuki '95

$$\star\beta^\Lambda = (\mathrm{Im}\mathcal{N}^{-1})^{\Lambda\Sigma}\alpha_\Sigma - \left((\mathrm{Im}\mathcal{N})^{-1}\mathrm{Re}\mathcal{N}\right)^\Lambda_{\ \Sigma}\beta^\Sigma \qquad \mathcal{N}_{\Lambda\Sigma} = \bar F_{\Lambda\Sigma} + 2\mathrm{i}\frac{(\mathrm{Im}Fz)_\Lambda (\mathrm{Im}Fz)_\Sigma}{(z\mathrm{Im}F\bar z)}$$

Alternative basis: $\{\Omega, \chi_a, \bar\chi_a, \bar\Omega\} \in H^3 = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3}$ $\qquad \chi_a(z) = \partial_{z^a}\Omega + \Omega\,\mathcal{K}_a$

$$C = \rho\,\Omega + \varrho^a \chi_a + \bar\varrho^a \bar\chi_a + \bar\rho\,\bar\Omega \;\longrightarrow\; J(C) = \mathrm{i}\left(\rho\,\Omega - \bar\rho\,\bar\Omega\right) + \frac{\mathrm{i}}{3}\left(\varrho^a\chi_a - \bar\varrho^a\bar\chi_a\right)$$

Relations: $\bar\rho = -\mathrm{i}e^{\mathcal{K}}\left(z^\Lambda\tilde\zeta_\Lambda - F_\Lambda\zeta^\Lambda\right) \qquad \mathrm{Im}\left(\varrho^a\partial_{z^a}Z_\gamma + (\rho + \varrho^a\mathcal{K}_a)Z_\gamma\right) = \mathscr{C}_\gamma$

Final result

$$a_\Lambda d\zeta^\Lambda + a^\Lambda d\tilde{\zeta}_\Lambda = -2\pi^3 \left[d\Theta_\gamma + 2\mathrm{i}\, \mathcal{C}_\gamma \right]$$

$$a_{z^a} dz^a + a_{\bar{z}^a} d\bar{z}^a = \mathrm{i}\pi^2 \mathrm{Re}\mathcal{T}_\gamma\, \bar{\partial} \log(e^{\mathcal{K}} \bar{Z}_\gamma)$$

$$a_\phi d\phi = \frac{\mathrm{i}\pi^2}{2} \mathrm{Re}\mathcal{T}_\gamma\, d\phi$$

$$ds^2_{\mathrm{inst}} = \sum_\gamma 2\pi e^\phi\, g_o\, \Omega_\gamma \left(\sum_{k=1}^{\infty} k^{-1/2}\, e^{-k\mathcal{T}_\gamma} \right) \left(\sum_m a_m\, d\lambda^m \right)^2 + \mathcal{O}(d\mathcal{T}_\gamma)$$

Perfectly coincides with the instanton corrected metric predicted by dualities!

Type IIB and mirror symmetry

Mirror symmetry

$\mathrm{IIA}/\mathfrak{Y} = \mathrm{IIB}/\tilde{\mathfrak{Y}}$

mirror CYs

$$\mathcal{M}_{\mathrm{HM}}^{\mathrm{IIA}}(\mathfrak{Y}) = \mathcal{M}_{\mathrm{HM}}^{\mathrm{IIB}}(\tilde{\mathfrak{Y}})$$

Two metrics are identical

But if we want to express the metric in terms of natural Type IIB fields (transforming simply under S-duality), we need *mirror map*

receives instanton corrections

D(-1), D1 – known; D3 – partially; D5 – make no sense without NS5

However, at leading order in g_s their effect is $\mathcal{O}(d\mathcal{T}_\gamma)$ → classical mirror map is sufficient

The resulting metric is exactly reproduced by a similar calculation of string amplitudes

New features:

- more branes to analyze
- existence of bound states of branes of different type
- Kähler geometry instead of complex structure moduli

Conclusions

- String field theory is able to fix all apparent divergences and ambiguities in instanton contributions to string amplitudes

- String amplitudes perfectly reproduce the results predicted by dualities, which provides a highly non-trivial test for both approaches

- There is a way to extract directly the *metric* on the moduli space and not only its *curvature*

Some open problems:

see Manki's talk

- Compute instanton corrections where dualities do not help
  (phenomenologically relevant $\mathcal{N} = 1$ string compactifications)

- Go beyond the leading order in the string coupling

- Get insights about *NS5-brane instantons* in CY compactifications which remain not fully understood