Timelike Liouville gravity

&

its sphere partition function

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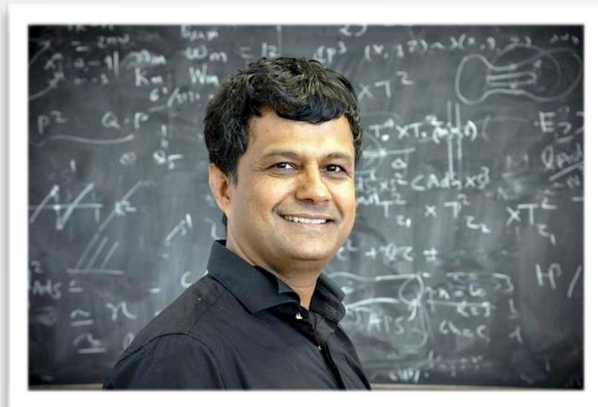
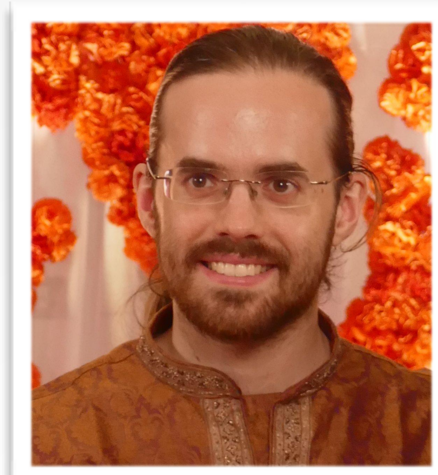
Together with



Atish Dabholkar



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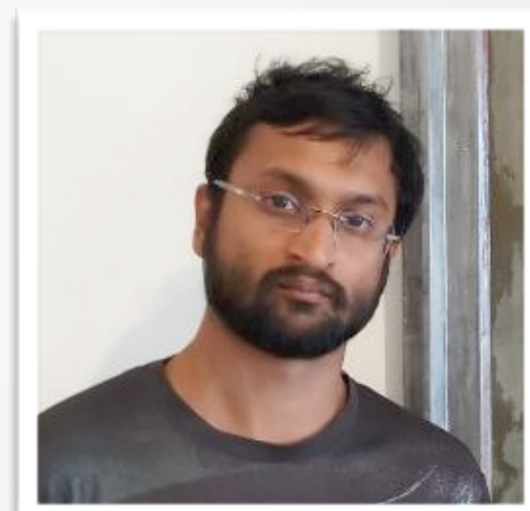


Aditya Bawane

Path integral for Gravity

- Consider the Euclidean path integral for gravity :

in the conformal gauge : $g_{\mu\nu} = e^{2\Omega} h_{\mu\nu}$ $\qquad \mathcal{Z} \sim \int \mathcal{D}\Omega \;\; e^{-\int \sqrt{h}\,\left(-(\nabla\Omega)^2 + \ldots + \mu\, e^{d\,\Omega}\right)}$

[Gibbons, Hawking,Perry]

- In **2d**, the metric reduces to the conformal factor.

- The effective action of 2d gravity coupled to matter CFT is the **Liouville** action

[Polyakov]

$$\mathcal{Z} \sim \int \mathcal{D}\Omega\, e^{-\int \sqrt{h}\left(-c_m (\nabla\Omega)^2 + \ldots + \mu\, e^{2\Omega}\right)}$$

1. **Timelike Liouville** reproduces the problematic minus sign
2. Is a **CFT**
3. Allows for coupling to **unitary matter**

Questions

1) Can timelike Liouville theory be a consistent model of Euclidean 2d **gravity with a cosmological constant ?**

- **Spectrum** of physical states & analytic continuation for finite **correlators**
- Time-dependent background in string theory [Strominger&Takayanagi; Sen;…]

2) Can we explicitly **compute** anything (finite) with its **path integral ?**

- Perturbative **S^2 path integral** & proposal for an all-orders result
- Finite on-shell action, classical entropy of black holes [Mahajan, Stanford & Yan;…]

Outline

1. Timelike Liouville gravity
2. Spectrum & analytic continuation
3. Sphere partition function
4. Outlook

1.- Timelike Liouville gravity

2d Gravity + matter CFT

- Euclidean path integral : $\mathcal{Z}_{(0)} = \int \frac{\mathcal{D}\, g_{\mu\nu}}{V_{\text{diff}}}\, \mathcal{D}\, X^i\, e^{-S_m[X^i, g] - \int \sqrt{g}\, \mu_0}$ spherical topologies

- Conformal gauge : $g_{\mu\nu} = e^{2\Omega} h_{\mu\nu}$

$$\frac{\mathcal{D}\, g_{\mu\nu}}{V_{\text{diff}}} = \mathcal{D}\Omega\, \mathcal{D}(b, c)\, e^{-S_{\text{gh}}[g,b,c]}$$

$$\mathcal{Z} = \mathcal{Z}_m\, \mathcal{Z}_{gh} \int \mathcal{D}\Omega\; e^{-\frac{q^2}{4\pi}\int \sqrt{h}\,\left(-(\nabla\Omega)^2 \,-\, R_h\,\Omega \,+\, 4\pi\mu\, e^{2\Omega}\right)}$$ [Polyakov]

$$q^2 = \frac{c_m + c_{gh} + 1}{6} = \frac{c_m - 25}{6}$$

[David; Distler & Kawai,..]

Semiclassical limit : $q^2 \to \infty$

Timelike Liouville gravity

- Rescaling as $\chi \equiv q\,\Omega$ we get the **Timelike Liouville** action

$$S_{TL}[\chi] = \frac{1}{4\pi}\int d^2x\,\sqrt{h}\,\left(-(\nabla\chi)^2 - q\,R_h\,\chi + 4\pi\mu\,e^{2\beta\chi}\right)$$

- Residual gauge invariance : $g_{\mu\nu} = e^{2\chi/q}h_{\mu\nu}$

$$\chi \to \chi - q\,\sigma(x)$$
$$h_{\mu\nu} \to e^{2\sigma(x)}h_{\mu\nu}$$

→ Conformal invariance on flat space

- The anomalous dimension of the cosmological constant operator, requires introducing the Liouville coupling $V_\beta = e^{2\beta\chi}$

$$q = \frac{1}{\beta} - \beta$$

[David; Distler&Kawai,..]

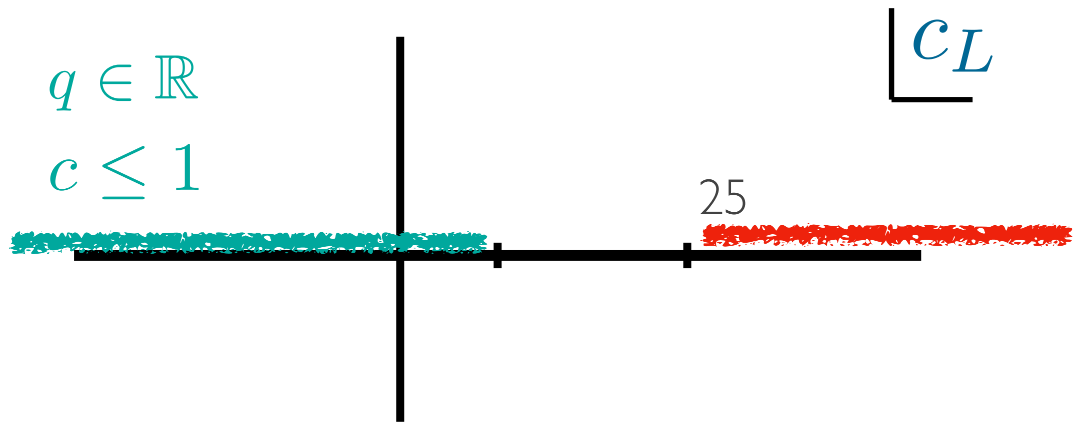
Semiclassical limit : $\beta^2 \to 0$

- Central charge : $c_L = 1 - 6q^2$ $\longrightarrow$ $c_L + c_m + c_{gh} = 0$

Timelike $q \in \mathbb{R}$

$c \leq 1$

[Polchinski; Cooper et al; Strominger & Gutperle; &Takayanagi; Martinec et al; Schomerus;…]



Spacelike

$c \geq 25$

$q \in i\mathbb{R}$

[Polyakov; Moore et al; Seiberg; Belavin et al; Zamolodchikov;…]

* Analytic continuation $\longrightarrow$ Spacelike Liouville

$$\chi = -i\varphi$$
$$q = -iQ$$
$$\beta = ib$$

$$S_{SL}[\varphi] = \frac{1}{4\pi}\int d^2x\,\sqrt{h}\,\left((\nabla\varphi)^2 + Q\,R_h\,\varphi + 4\pi\mu\,e^{2b\varphi}\right)$$

$\rightarrow$ Does not work at the quantum level

2.- Spectrum & analytic continuation

Spectrum from bootstrap

- Primaries labeled by Liouville charge : $V_\alpha = e^{2\alpha\chi}$ $\Delta = \alpha(q+\alpha)$ [David; Distler&Kawai; Polchinski; Seiberg; D'Hoker et al.; KPZ;…]

  Parametrize as $\alpha = -\frac{q}{2} + i\,E$

¿ What is the range of energies ?

- Conformal bootstrap answer : from **crossing symmetry** of $C_4 = \int dE_s\, C_3\, C_3\, |\mathcal{F}_{E_s}|^2$

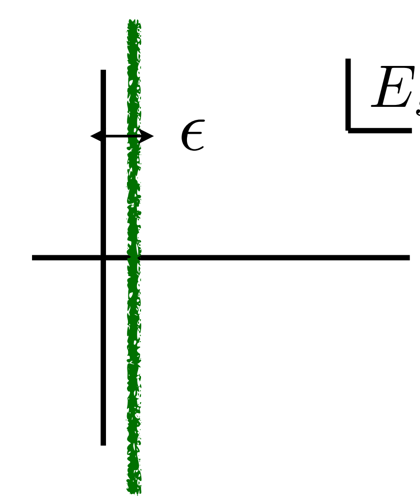
$$C_3(\alpha_1,\alpha_2,\alpha_s) = \frac{2\pi}{\beta}\left(\pi\mu\gamma(-\beta^2)\beta^{2+2\beta^2}\right)^{-\frac{q+\alpha_t}{\beta}} \frac{\Upsilon(\beta-q-\alpha_t)\prod_{i=1}^{3}\Upsilon(\beta+2\alpha_i-\alpha_t)}{\Upsilon(\beta)\prod_{i=1}^{3}\Upsilon(\beta-2\alpha_i)}$$

[Kostov & Petkova; Zamolodchikov; Schomerus]

The integrand
- has poles on the imaginary E_s axis
- but diverges at $E_s \to \pm\infty$

→ contour $E_s \in i\,\mathbb{R} + \epsilon$ gives a crossing-symmetric 4-point function

[Ribault & Santachiara]



¿ Is this the spectrum of 2d timelike Liouville **gravity** ?

Spectrum from BRST cohomology

- **Generalize** the BRST cohomology for **spacelike** Liouville to timelike Liouville

[Bouwknegt, McCarthy & Pilch; Mukherji et al; Mukhi; Itoh & Ohta; Bilal; Lian & Zuckerman]

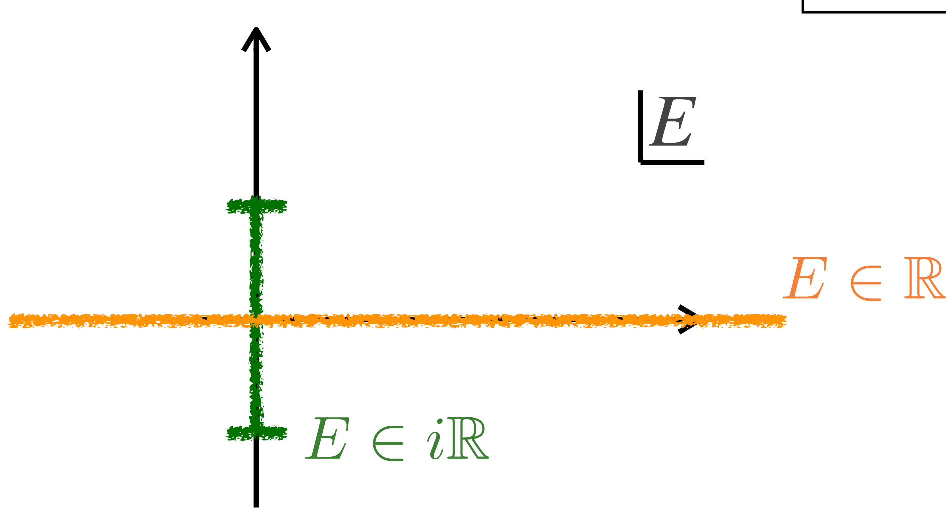
- **Main idea** : states in the BRST cohomology are in one-to-one with states in the light cone gauge, which are manifestly positive-norm

A general state : $|\psi\rangle = \prod_{m>0} (\alpha^{\chi}_{-m})^{N^{\chi}_m} (\alpha^{X}_{-m})^{N^{X}_m} (b_{-m})^{N^{b}_m} (c_{-m})^{N^{c}_m} |E, P, \downarrow\rangle \otimes |\psi_\perp\rangle$

States in the cohomology satisfy* :

$$\left(N^{\chi} + N^{X} + N^{b} + N^{c}\right) |\psi\rangle = 0 \longrightarrow |\psi\rangle = |E, P, \downarrow\rangle \otimes |\psi_\perp\rangle$$

On-shell condition : $L_0 = 0 \longrightarrow$ $\boxed{L_0^{\perp} + \frac{Q^2}{4} - \frac{q^2}{4} + P^2 - E^2 - 1 = 0}$



[TB, A. Dabholkar, H. Erbin]
[TB, H. Erbin, M. Kudrna]

No ghosts and gravity spectrum

1. The cohomology includes some discrete states, with ghost excitations

$$r > 0 : \quad (\alpha^{+}_{-r})^{u}\,(c_{-r})^{v}\,|E,P,\downarrow\rangle \otimes |\psi_{\perp}\rangle$$
$$r < 0 : \quad (\alpha^{-}_{r})^{u}\,(b_{r})^{v}\,|E,P,\downarrow\rangle \otimes |\psi_{\perp}\rangle$$

however such states require $P \in i\mathbb{R}$ $\longrightarrow$ ruled out by **Hermiticity** of the matter field X

No negative-norm states in the spectrum

2. Bootstrap and BRST spectra are different :

- Bootstrap : $E \in i\mathbb{R}$
- Gravity : $E \in \mathbb{R}, i\mathbb{R}$

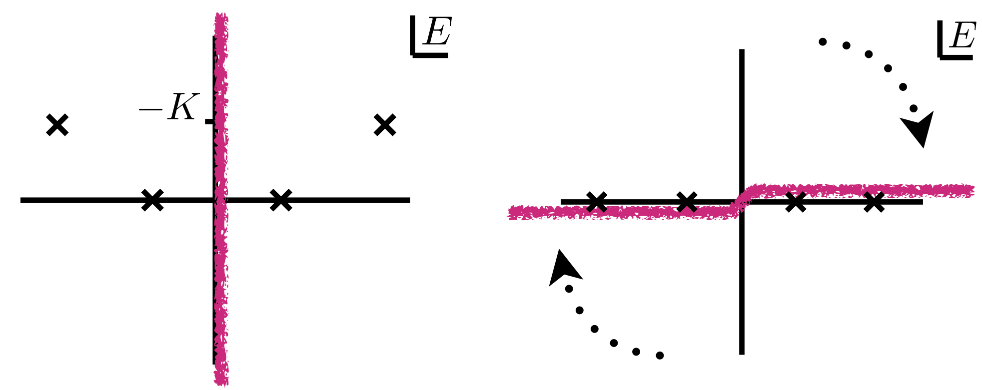
¿ What is the prescription to **analytically continue** the bootstrap 4-point function to real energies **?**

Analytic continuation

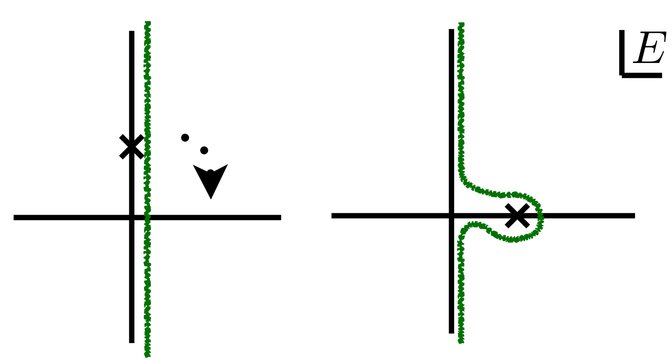
Want to continue the 4-point function to real energies

$$C_4(z_i, E_i) = \int dE_s\, C_3(E_1, E_2, E_s)\, C_3(-E_s, E_3, E_4) |\mathcal{F}_{E_s}(E_i, z_i)|^2$$

But integrand diverges at $E_s \to \pm\infty$ as $|q|^{-2E_s^2}$
no usual Wick rotation due to the contribution
from the arcs at infinity



- Prescription : 1. define correlators for imaginary external and internal Energies
  2. analytically continue the external energies, while
     a) deforming the contour smoothly to avoid poles crossing it
     b) keeping the contour ends fixed at $E_s \to \pm i\infty$



Prescription from String Field Theory
[Pius & Sen]

Generalizes the usual Wick rotation when integrand diverges

Analytic continuation

Continue the **external** energies to **real**, and check the movement of the poles:

$$C_4(z_i, E_i) = \int dE_s \, C_3(E_1, E_2, E_s) \, C_3(-E_s, E_3, E_4) |\mathcal{F}_{E_s}(E_i, z_i)|^2$$

- The **poles** of the integrand **do not depend** on the **external energies !**

$$C_3(\alpha_1, \alpha_2, \alpha_s) \sim \frac{\Upsilon(\beta - q - \alpha_t) \prod_{i=1}^{3} \Upsilon(\beta + 2\alpha_i - \alpha_t)}{\Upsilon(\beta) \prod_{i=1}^{3} \Upsilon(\beta - 2\alpha_i)}$$

zeros at $\alpha_i \in -\frac{q}{2} \pm \frac{1}{2}\left(\frac{\mathbb{N}}{\beta} + \mathbb{N}\,\beta - \{0\}\right)$

The simple analytic continuation of the external energies of the Ribault-Santachiara correlator gives a 4-point function that is crossing-symmetric on all the physical states of the BRST

[TB, A. Dabholkar, H. Erbin]

Summary

- Gravitational spectrum of real energies and free of negative-norm states
- Consistent theory of 2d quantum gravity, at the level of un-integrated correlators

3.- Sphere partition function

Sphere saddle

Timelike Liouville on the two-sphere fiducial:

$$S_{TL}[\chi] = \frac{1}{4\pi}\int d^2x\,\sqrt{h}\,\left(-(\nabla\chi)^2 - q\,R_h\,\chi + 4\pi\mu\, e^{2\beta\chi}\right) \qquad \text{with} \qquad ds_h^2 = 4\upsilon\,\frac{dzd\bar{z}}{(1+z\bar{z})^2}$$

EoM : $\quad -\nabla_h^2\chi = -\frac{q}{\upsilon} + 4\pi\beta\mu\, e^{2\beta\chi}$

Real **constant solution** : $\quad \chi_c = \frac{1}{2\beta}\log\left(\frac{q}{4\pi\beta\mu\upsilon}\right) \longrightarrow$ **'physical' sphere**

[Polchinski]

$\mu > 0 \quad \& \quad c_m > 25$

$$ds_g^2 = \frac{q}{\pi\beta\mu}\frac{dzd\bar{z}}{(1+z\bar{z})^2}$$

Gauge fixing

Back to the path integral for 2d gravity :

Saddle-point approx. & 1st correction in $\frac{1}{q^2} \sim \beta^2$

$$\mathcal{Z} = \mathcal{Z}_{m+gh}[v] \int \frac{\mathcal{D}\chi}{\mathrm{vol}_{PSL(2,\mathbb{C})}} e^{-S_{tL}[\chi_c+\chi]}$$

Residual gauge invariance, but infinite volume: fix the gauge further, down to

$$\mathrm{vol}_{SO(3)} = 2\pi^2$$

Further gauge fixing :

given $\chi = \sum_{l,m} \chi_{lm} Y_{lm}(\theta,\phi)$, fix $\chi_{1m} = 0, \qquad m = -1,0,1$ [Distler & Kawai]

$$\Delta_{FP} = \det \frac{\delta \chi_{1m}}{\delta \alpha_n} = -\frac{16}{3\sqrt{3}} \pi^{3/2} q^3 \left(1 - \frac{27}{20\pi} \frac{1}{q^2} \sum_m \chi_{2m}^2 + \mathcal{O}(\frac{1}{q^3})\right)$$

Integral

$$\frac{e^{-S_{tL}[\chi_c]}}{2\pi^2}\int \mathscr{D}\chi \;\; \Delta_{FP}(\beta^2\chi^2_{2m}) \;\; \delta\left(\chi_{1m}\right) \;\; e^{-\frac{1}{4\pi}\int -(\nabla\chi)^2+\frac{2}{v}(1-\beta^2)\chi^2+\frac{4}{3v}\beta\chi^3+\frac{2}{3v}\beta^2\chi^4+\ldots}$$

1.- Dependence on **cosmo constant** only in the saddle point contribution : $\mathscr{Z}(\mu) = \mu^{-q/\beta}\;\mathscr{Z}(1)$

2.- No dependence on **radius** in the χ action : $\mathscr{Z}(v) = v^{c/6}\,\mathscr{Z}(1)$ $\;\rightarrow$ anomaly cancellation!

3.- Rotation of the integration contour : $\chi \rightarrow \pm i\,\chi$ [alla Gibbons, Hawking, Perry]

4.- Quadratic integral : $\int \prod_{l,m} d\chi_{lm}\;\Delta_{FP}(\beta^2\,\chi^2_{2m})\;\prod_{0,\pm1}\delta(\chi_{1m})\;e^{-\frac{1}{4\pi}\sum_{l,m}[(l(l+1)-2(1-\beta^2)]\,\chi^2_{lm}}$

- l=0 mode : Wick rotate back $\pm i$
- l=1 modes : almost zero modes!
- l=2 modes : also in the Fadeev-Popov determinant

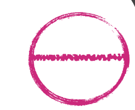
5.- Interaction terms :

Double-tadpole and cactus are logarithmically divergent, but exactly cancel each other!

 $= -\frac{2}{3}\sum_{\{l_i\neq 1\}}\prod_{i=1,2,3}\frac{2l_i+1}{l_i(l_i+1)-2}\begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0\end{pmatrix}^2$ Wigner 3-j symbol

Partition function

$$\frac{\mathcal{Z}}{\mathcal{Z}_{m+gh}} = \pm i\, C \left(\frac{\mu}{\Lambda_{uv}^2}\right)^{-\frac{q}{\beta}} \frac{e^{-\frac{1}{\beta^2}\log(4\pi e \beta^2)}}{\beta} \left\{1 - \left(\frac{1}{6} + \frac{27}{8} + 2\gamma_E + \log 4\pi - \right.\right.$$



$$\left.\left.\right)\beta^2 + \dots\right\}$$

[D. Anninos, TB, B. Mühlmann]

- Timelike Liouville gravity with unitary matter CFT enjoys a finite semiclassical expansion around the sphere saddle
- The Wick rotation of the integration contour is a sensible prescription, overall phase but real corrections
- TL gravity has the CFT-type radius anomaly from the Liouville sector. But overall anomaly cancellation consistent with gravity
- Diffeos gauge fixing contributes to the leading β-dependence and to the semiclassical corrections, and in a way consistent with conformal invariance
- Suggests it is finite to all orders [B. Mühlmann]

Partition function to all orders ?

We can compare with the analytic continuation of the *spacelike* Liouville partition function obtained with the conformal bootstrap :

$$\lim_{\beta\to 0} \mathcal{Z}_{sL,nonpert}(b=-i\beta) = i\left(1-e^{2i\pi/\beta^2}\right)\mathcal{Z}_{tL,pert}(\beta)$$

2n complex saddle

$\tilde{\chi}_c = \chi_c + i\pi/\beta$ also solution

Proposal : $\mathcal{Z}_{(0)} = \pm\, \mathcal{Z}_{m+gh}\left(\pi\mu\gamma(-\beta^2)\right)^{-q/\beta} e^{q^2(1-\log 4)} \dfrac{1+\beta^2}{q\,\gamma(-\beta^2)\,\gamma(-\beta^{-2})}$

Critical strings : $\beta = 1$ (and $\mu = 0$), and in this case $\mathcal{Z}_{(0)} = 0$

4.- Summary & Outlook

Summary

- Timelike Liouville gravity with $\mu > 0$ & coupled to matter CFT $c_m > 25$ has a finite sphere partition function

- Diffeomorphism gauge fixing contributes both to the overall β-power of the path integral and to the semiclassical corrections, and in a way consistent with conformal invariance

Outlook

- Check the proposal for g=0 partition function.
- Role of the complex saddle, and why only n=1 ?
- Lessons for Lorentzian signature ?
- Lessons for the microscopics ? $S \sim -\frac{q}{\beta} \log \mu = -q^2 (1 + \frac{1}{q^2} - \frac{1}{q^4} + \ldots) \log \mu$
- Timelike Liouville on the disk & study of its brane solutions

Thanks a lot