The annulus diagram with ZZ boundaries

(SFT @ Prague)

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Introduction

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Z &= Z^{(0)} + Z^{(1)} + \dots && (1) \\
&= \exp\left(g_s^{-2}\, c_0 + g_s^0\, c_1 + g_s^2\, c_2 + \dots\right) + Z^{(1)} + \dots && (2)
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\log Z &= \log Z^{(0)} + \frac{Z^{(1)}}{Z^{(0)}} + \dots && (3)\\
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\frac{Z^{(1)}}{Z^{(0)}} =: e^{-T}\, \mathcal{N}\, (1 + O(g_s)) \tag{4}
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Goal: Compute $\mathcal{N}$, or more precisely $\mathcal{N}\, T^{\frac{1}{2}}$.

The story of this project-1

`ZZ Instantons and the Non-Perturbative Dual of c = 1 String Theory`

`Bruno Balthazar, Victor A. Rodriguez, Xi Yin`

`1907.07688`

These authors computed the 2-point function of closed strings to order $g_s \exp\left(-g_s^{-1}\right)$.

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`Fixing an Ambiguity in Two Dimensional String Theory Using String Field Theory`

`Ashoke Sen`

`1908.02782`

The story of this project-2

1. [Saad-Shenker-Stanford] JT gravity as a matrix integral. JT gravity is a particular limit of noncritical string theory. In this limit the Liouville sector is semi-classical, there is a large coefficient in front of its action.

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2. The ZZ instanton in this context give rises to the transition between the "ramp" and "plateau" features in the spectral form factor, at extremely long times of order e^S. The one-loop factor around the ZZ instanton is ill-defined.

3. Douglas Stanford suggested to me that the techniques that Ashoke was developing for dealing with instantons in the $c = 1$ system might be relevant to this problem.

Collaborators

Chitraang Murdia, PhD @ U.C. Berkeley

Applying for postdocs this fall

Collaborators

Dan Stefan Eniceicu
3rd year PhD @ Stanford University

Collaborators

Prof. Ashoke Sen
ICTS Bangalore

The strength of non-perturbative effects

[David-Shenker-Ginsparg-ZinnJustin-Polchinski] A very important point is that the tension or the action of the instanton scales as g_s^{-1} rather than g_s^{-2}. Related to the $(2g)!$ growth of the coefficients in the perturbation series (as opposed to $g!$).

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As we will see, $\mathcal{N} \propto g_s^{\frac{1}{2}}$, so the combination $\mathcal{N}\, T^{\frac{1}{2}}$ is independent of g_s and can be compared to a dual description.

The worldsheet of the minimal string

The worldsheet theory consists of

1. The (p', p) minimal model, which is two-dimensional CFT (in general it is non-unitary). This can be thought of as the matter sector and has $c = 1 - \frac{(p-p')^2}{4pp'} < 1$. The simplest case $(2, 3)$ corresponds to an empty matter sector, which has $c = 0$.

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2. The Liouville CFT, which is a remnant of the conformal mode of the worldsheet metric. The Liouville sector has $c > 25$.
3. The bc-ghost CFT with $c = -26$.

Duality with matrix integrals

[Brezin,David,Distler,Douglas,Gross,Kawai,Kazakov,Knizhnik,Migdal,Polyakov, Shenker,Zamolodchikov^2 and many others]

When $(p', p) = (2, p)$, the above string theory is dual to an integral over one Hermitian matrix. The integral in taken to be in the large-N and double-scaling limit.

$$Z(N, t, g_k) := \int \frac{\mathrm{d}^{N^2} M}{\mathrm{vol}(U(N))} \exp\left[-\frac{N}{t} \operatorname{Tr} V(M)\right] \tag{6}$$

$$V(x) = \frac{g_2}{2} x^2 + \frac{g_3}{3} x^3 + \ldots \tag{7}$$

M is an $N \times N$ Hermitian matrix.

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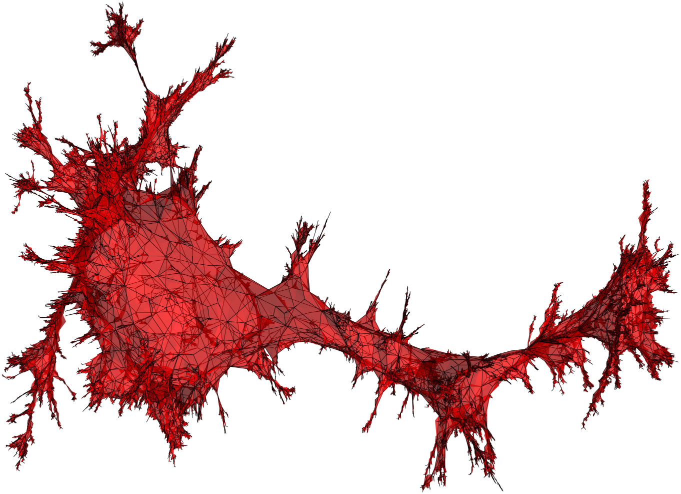
If $p' \geqslant 3$, the dual theory consists of an integral over two Hermitian matrices, with action
$\operatorname{Tr}(V_1(M_1) + V_2(M_2) + M_1 M_2)$.

Random surfaces and double-line Feynman diagrams

Where does this duality come from?



Picture Credit: Jeremie Bettinelli

The matrix computation

The quartic matrix integral

$$Z(N,t,g_2,g_4) := \int \frac{\mathrm{d}^{N^2} M}{\mathrm{vol}(U(N))} \exp\left[-\frac{N}{t}\operatorname{Tr} V(M)\right] \tag{8}$$

$$= \frac{1}{N!}\int \prod_{i=1}^{N} \frac{\mathrm{d}x_i}{2\pi}\, \Delta(x)^2 \exp\left(-\frac{N}{t}\sum_i V(x_i)\right) \tag{9}$$

$$V(x) = \frac{g_2}{2}x^2 + \frac{g_4}{4}x^4 \tag{10}$$

$$\Delta(x) := \prod_{i<j}(x_i - x_j) \tag{11}$$

We will be interested in the case with $g_2 > 0$ and $g_4 < 0$.
We would like to evaluate the integral by saddle point (N is large).

The quartic matrix integral

For this, it is convenient to introduce the density of eigenvalues $\rho(x)$ and the resolvent $R(x)$.

$$\rho(x) := \frac{1}{N} \langle \mathrm{Tr}\, \delta(x - M) \rangle \tag{12}$$

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$$\omega_0(x) = \frac{1}{2t} \left(g_2 x + g_4 x^3 - (a_2 x^2 + a_0) \sqrt{x^2 - b^2} \right), \text{ with} \tag{14}$$

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$$a_0 = g_2 + g_4 b^2/2 \tag{16}$$

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Some comments on the solution

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3. The final equation is a non-linear polynomial equation that determines b.

The quartic matrix integral

We want to go beyond the perturbative expansion of $\log Z$ and include effects from one-eigenvalue instantons.
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A single eigenvalue in the matrix integral at position x_i feels an effective potential

$$V(x_i) - \frac{2t}{N} \sum_{j:j\neq i} \log|x_i - x_j| \,. \tag{22}$$

It is useful to introduce a *holomorphic* effective potential

$$V_{\text{eff}}(x,t) := V(x) - 2t \int_{-b}^{b} \mathrm{d}y\, \rho(y) \log(y-x) \,. \tag{23}$$

The actual potential felt by an eigenvalue is the *real part* of the holomorphic effective potential.

More general potential

In a more general polynomial potential, we have the following important relations

$$V'_{\text{eff}}(x) = M(x)\sqrt{x^2 - b^2}\,, \tag{24}$$

$$\rho(x) = \frac{1}{2\pi t} M(x)\sqrt{b^2 - x^2}\;\Theta(b - |x|)\,, \tag{25}$$

where $\Theta(x)$ is the Heaviside step function and $M(x)$ is a polynomial determined from the potential by the requirement that the resolvent $\omega_0(x) = \frac{1}{2t}(V'(x) - M(x)\sqrt{x^2 - b^2})$ behaves as $1/x$ as $x \to \infty$ on the physical x-sheet. (The endpoint b is also determined from the same requirement.)

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The one-eigenvalue instantons correspond to the extrema of $V_{\text{eff}}(x)$ with x being outside the interval $[-b, b]$. In other words, the one-eigenvalue instantons are the zeroes of $M(x)$.

Double scaling in the quartic matrix integral

Recall the equation governing b^2

$$4g_2 r + 3g_4 r^2 = 16t\,, \quad r := b^2 \tag{26}$$

Since $g_2 > 0$ and $g_4 < 0$, the LHS has a maximum value at $r = 2g_2/(-3g_4)$.

So if $t > g_2^2/(-12g_4)$ then we do not have a real solution for b. This is one way of getting at the double-scaling limit.

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One other thing that happens at the double-scaling limit is that the planar free energy F_0 hits a singularity as a function of g_4. So this value of g_4 defines the radius of convergence of the g_4-perturbation series (if we were to compute the planar diagrams using perturbation theory in g_4).

Double scaling in the quartic matrix integral

So planar graphs with more and more vertices become important, and so Feynaman diagrams / the 2d random quadrangulated surfaces begin to approach a continuum limit.

It is in this limit that the quartic matrix integral is dual to the $(2, 3)$ minimal string.

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It is in this limit that the quartic matrix integral is dual to the $(2, 3)$ minimal string.

For the $(2, p)$ minimal string with $p \geqslant 3$ an odd integer, one can start with an even potential of degree $p + 1$. Now there are various possible double scaling limits, and we would like to focus on the one which is dual to the standard Liouville Lagrangian with only the cosmological constant term $\mu\, e^{2b\phi}$ in the action.[Moore-Seiberg-Staudacher]

The conformal background

One can compute the leading order density of eigenvalues from Liouville theory, and then tune the coefficients of the matrix potential so that the density of eigenvalues matches. This specific density is given by

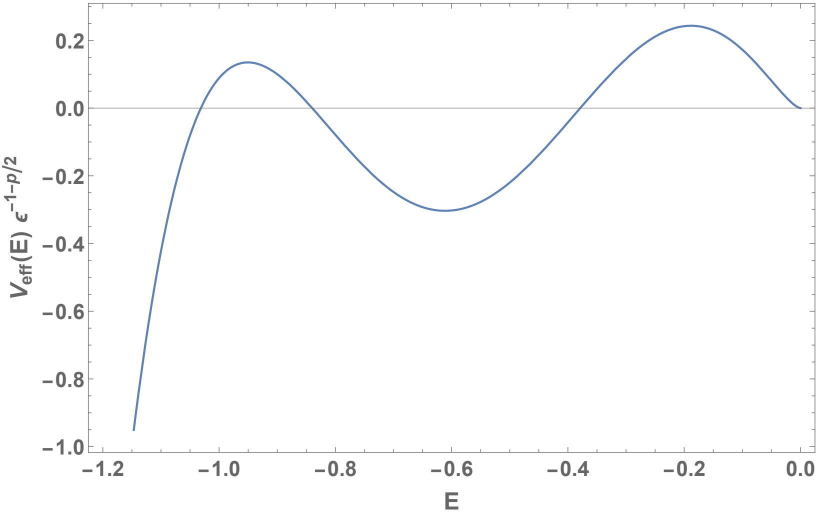
$$x =: -b + \varepsilon\, E \tag{27}$$

$$\langle \rho(E) \rangle^{(0)} = \frac{e^{S_0}}{\pi} \sinh\left(p \,\mathrm{arcsinh} \sqrt{\frac{E}{2\kappa}} \right) \Theta(E)\,. \tag{28}$$

$$e^{S_0} := N\, \varepsilon^{\frac{p}{2}+1} \tag{29}$$

Double scaling means taking the limit $\varepsilon \to 0$ and $N \to \infty$ keeping e^{S_0} fixed.

The effective potential for $p = 7$



Extrema of the effective potential

Let us now look at the extrema of the one-eigenvalue effective action. As we move towards negative energies starting at $E = 0$, the first zero of $V'_{\text{eff}}(E)$ occurs at

$$E^\star = -2\kappa \sin^2 \frac{\pi}{p} \,. \tag{30}$$

We record the values of $V_{\text{eff}}(E^\star)$ and $V''_{\text{eff}}(E^\star)$:

$$V_{\text{eff}}(E^\star) = e^{S_0} \kappa \, \frac{4p \sin(2\pi/p)}{p^2 - 4} \,, \tag{31}$$

$$V''_{\text{eff}}(E^\star) = -e^{S_0} \kappa^{-1} \frac{p}{\sin(2\pi/p)} \,. \tag{32}$$

Instanton correction to the free energy

Now we organize various contributions to Z depending on how many eigenvalues are in the classically allowed region $E > 0$ and how many are in the classically forbidden region $E < 0$.

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The leading contribution $Z^{(0)}$ comes from the integration region where all eigenvalues are in the classically allowed region.

The next important contribution $Z^{(1)}$ comes from the integration region when only one eigenvalue is in the forbidden region.

$$Z = Z^{(0)} + Z^{(1)} + \dots \tag{33}$$

$$\log Z = \log Z^{(0)} + \frac{Z^{(1)}}{Z^{(0)}} + \dots \tag{34}$$

$$\frac{Z^{(1)}}{Z^{(0)}} = e^{-T}\,\mathcal{N}\,(1 + O(g_s)) \tag{35}$$

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$$\frac{Z^{(1)}}{Z^{(0)}} = \exp\left(-V_{\mathrm{eff}}(E^\star)\right) \times \frac{i}{2}\sqrt{\frac{2\pi}{|V''_{\mathrm{eff}}(E^\star)|}} \times \frac{1}{-8\pi E^\star} \qquad (36)$$

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Comparing this to $\mathcal{N}\, e^{-T}$ we get

$$T = V_{\text{eff}}(E^\star) = e^{S_0}\kappa\, \frac{4p\sin(2\pi/p)}{p^2-4}\,, \tag{37}$$

$$\mathcal{N} = e^{-\frac{S_0}{2}}\kappa^{-\frac{1}{2}}\, \frac{i}{16\sqrt{\pi}}\sqrt{\frac{\cos(\pi/p)}{p\sin^3(\pi/p)}}\,. \tag{38}$$

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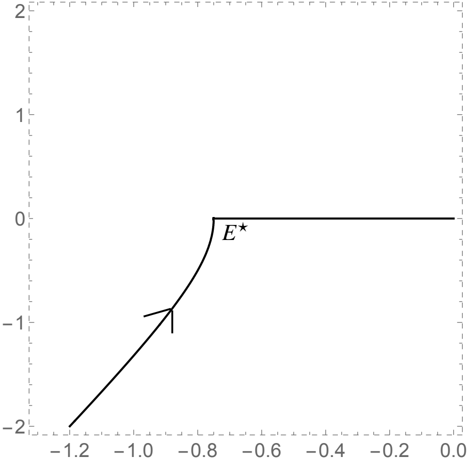
It is natural to factor out $T^{-\frac{1}{2}}$ from the expression for $\mathcal{N}$, and so we write the above result as

$$\boxed{\mathcal{N} = T^{-\frac{1}{2}}\, \frac{i}{\sqrt{32\pi}}\, \frac{\cot(\pi/p)}{\sqrt{p^2-4}}} \tag{39}$$

Contour of integration in the E plane



Questions?

We would like to reproduce this result from the string description.

Recall: worldsheet theory contains the $(2, p)$ minimal model, Liouville, and bc-ghosts.

Liouville theory primer

A lot of things are known about the Liouville CFT.

$$Z_L = \int D\phi \, \exp(-I) \tag{40}$$

$$I = \int \mathrm{d}^2x \, \sqrt{g} \left(\frac{1}{4\pi} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{4\pi} Q R \phi + \mu \, e^{2b\phi} \right) \tag{41}$$

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2. Vertex operators $e^{2\alpha\phi}$. DOZZ formula for three point functions on the sphere.

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$$I = \int \mathrm{d}^2x \sqrt{g} \left(\frac{1}{4\pi} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{4\pi} Q R \phi + \mu \, e^{2b\phi} \right) \tag{41}$$

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Liouville theory primer

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3. FZZT boundary conditions, labeled by μ_{B}
4. ZZ boundary conditions, which carry a discrete set of labels

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$$\mathcal{N} = \text{exponential of the empty annulus} \tag{44}$$

The annulus partition function

Let t be the time in the open string channel and let $q = e^{-2\pi t}$. The partition function of Liouville theory on the cylinder with $(1, 1)$ ZZ boundary conditions on both ends is [Zamolodchikov^2]

$$Z_{\text{Liouville}}(t) = \left(q^{-1} - 1\right) q^{-\frac{(p-2)^2}{8p}} \eta(it)^{-1} \,. \tag{45}$$

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Multiplying the contribution $\eta(it)^2$ from the ghosts, we find

$$F(t) := \left(e^{2\pi t} - 1\right) \sum_{k=-\infty}^{\infty} \left(e^{-2\pi t k(2pk+p-2)} - e^{-2\pi t(pk+1)(2k+1)} \right). \tag{47}$$

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The quantity we are looking for is:

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It is important to note that the leading terms in $F(t)$ as $t \to \infty$ are the ones with $k = 0$:

$$F(t) = \left(e^{2\pi t} - 1\right)\left(1 - e^{-2\pi t} + O(e^{-4\pi t})\right) = e^{2\pi t} - 2 + O(e^{-2\pi t})\,.$$

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So we have (a) an open string tachyon and (b) two fermionic zero modes, which make the $t = \infty$ end of the integral divergent. The $t = 0$ end of the integral is convergent since these theories do not have a closed-string tachyon.

Table of the relevant states

State	L_0	Ghost no.	In Siegel gauge?	Field name	Grassmann parity
$c_1\|0\rangle$	-1	1	Yes	ϕ_1	even
$c_0c_1\|0\rangle$	-1	2	No	-	odd
$\|0\rangle$	0	0	Yes	p_1	odd
$c_0\|0\rangle$	0	1	No	ψ	even
$c_1c_{-1}\|0\rangle$	0	2	Yes	q_1	odd
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The moral of the story is "un-Faddeev-Popov-ing"

$$\mathcal{N} \propto \int dp_1 dq_1 \quad \longrightarrow \quad \frac{\int d\psi\, e^{-\psi^2}}{\int d\theta} = \frac{\sqrt{\pi}}{\int d\theta} \quad \text{[Sen]} \qquad (49)$$

Details of the computation

1. What would have been a worldvolume $U(1)$ gauge symmetry is not actually a local gauge symmetry, because the worldvolume is zero-dimensional. The gauge group volume $\int \mathrm{d}\theta$ is thus finite. [Sen]

$$\int d\theta = \frac{2\pi}{g_o} \tag{50}$$

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2. The open string coupling constant is related to the instanton action via T [Sen, Schnabl]

$$T = \frac{1}{2\pi^2 g_o^2} \tag{51}$$

Details of the computation

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\begin{aligned}
F(t) &= \left(e^{2\pi t} - 1\right) \sum_{k=-\infty}^{\infty} \left(e^{-2\pi t k(2pk+p-2)} - e^{-2\pi t(pk+1)(2k+1)}\right) \\
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\begin{aligned}
\mathcal{N} &= \left(\frac{\prod_f \hat{h}_f}{\prod_b h_b}\right)^{\frac{1}{2}} = \frac{\prod'_f \hat{h}_f}{\prod_b h_b^{1/2}} \\
&= \int \prod_b \frac{d\phi_b}{\sqrt{2\pi}} \prod_f{}' dp_f dq_f \, \exp\left[-\frac{1}{2}\sum_b h_b \phi_b^2 - \sum_f{}' \hat{h}_f p_f q_f\right]
\end{aligned}
$$

Details of the computation

$$\mathcal{N} = (\phi_1 \text{ integral}) \times \frac{(\psi \text{ integral})}{2\pi/g_o} \times \text{all other integrals} \tag{54}$$

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$$\boxed{\mathcal{N} = T^{-\frac{1}{2}}\,\frac{i}{\sqrt{32\pi}}\,\frac{\cot(\pi/p)}{\sqrt{p^2-4}}} \tag{56}$$

Questions?

Extensions of the result

In 2206.13531, we have extended this result to the general (p', p) minimal string theory.

The dual theory is a two-matrix model.

We take a general configuration of multiple instantons: a number ℓ_α of instantons of type α. The one-loop quantity matches between the string computation and the two-matrix integral.

The group whose volume we need to divide by is the unitary group $U(\ell)$.

Ongoing work

In a previous paper 2012.11624, Ashoke Sen found a mismatch between two computations of the annulus one-point function in $c = 1$ string theory:
(a) One was a matrix computation
(b) The other was a string theory computation which combined a numerical integration over moduli space and string field theory. [Balthazar,Rodriguez,Yin; Sen]

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(b) The other was a string theory computation which combined a numerical integration over moduli space and string field theory. [Balthazar,Rodriguez,Yin; Sen]

In upcoming work together also with Pronobesh Maity, we have managed to solve this puzzle. We computed the annulus one-point in $c < 1$ string theories, which does not require numerical integration over moduli space. Going to this simpler model allowed us to isolate a subtle issue in the SFT analysis of 2012.11624. The results now match.

Collaborators

Pronobesh Maity, PhD @ ICTS Bangalore

Applying for postdocs this fall

Thank you