From Fields *to* Strings

Pronobesh Maity

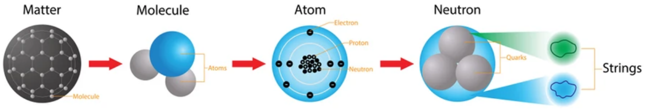
Based on (Gaberdiel-Gopakumar-Knighton-PM - arXiv:2011.10038) and (Bhat-Gopakumar-PM-Radhakrishnan - *arXiv:2112.05115*)

September 15, 2022

Perspective

Dominant viewpoint in the pre-AdS/CFT era: Strings as the ultimate building blocks of nature

"Reductionism"



[Image taken from Google]

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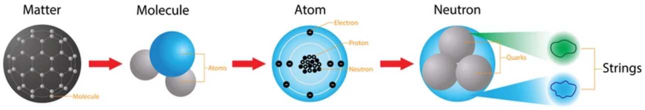


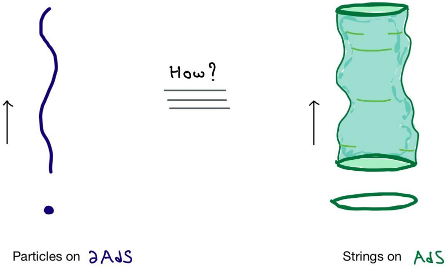
[Image taken from Google]

A different perspective after AdS/CFT: Strings as emergent geometry from Gauge theories (at large N).

"Emergence"

Motivation

How exactly do quantum field theories organise themselves into string theories in the large N limit? ['t Hooft'74]



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Maldacena's AdS/CFT concretizes such dualities, but both sides (boundary and bulk) are not decipherable simultaneously in the proposed examples, making it hard to portray the mechanism behind it. [Maldacena'97]

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Can we revisit Gopakumar's construction in the context of these exact dualities?

Motivation: Output

We will show how Gopakumar's prescription works explicitly for AdS_3/CFT_2 and (*to some extent*) for AdS_5/CFT_4:

Feynman Graph $\Leftrightarrow$ Strebel Graph $\Leftrightarrow$ Point in the String moduli space.

Proposal $\Longrightarrow$ Explicit Construction

Outline

1. Symmetric Product CFTs and Covering surfaces, and Matrix model,
2. Feynman graph $\Leftrightarrow$ Matrix model solution $\Leftrightarrow$ Strebel graph $\Leftrightarrow$ point in the String moduli space,
3. Twistor covering in AdS_3,
4. Twistor covering in AdS_5,
5. An explicit covering map for AdS_5 whose area reproduces Feynman propagator.

Symmetric Product Orbifold CFT

We will consider the symmetric product orbifold CFT

$$\mathrm{Sym}^K(\mathcal{M}) \equiv (\mathcal{M})^{\otimes K}/S_K$$

which corresponds to taking K copies of sigma models on $\mathcal{M}$ and identifying them by the permutation group S_K.

Untwisted sector: states invariant under the permutation.

Twisted sector: Single-cycle twist field σ_w (w is the cycle length)

$$\phi_I(e^{2\pi i}z+\zeta)\mathcal{O}_w(\zeta) = \phi_{\pi_w(I)}(z+\zeta)\mathcal{O}_w(\zeta)$$

Twist operator $\mathcal{O}_w$ induces a w-fold cyclic permutation amongst some of the K copies of the seed theory.

Twist Correlators and Covering Map

$$G = \langle \mathcal{O}_{w_1}(x_1) \cdots \mathcal{O}_{w_n}(x_n) \rangle$$

Locally the w-cycle twist field $\mathcal{O}_w(x_0)$ induces a w-fold covering

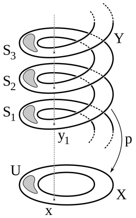
$$z \mapsto \Gamma(z) = x_0 + a(z - z_0)^w + \cdots$$

We can combine these local coverings near all the insertion points into an (auxiliary) global covering surface Σ described by the covering map

$$\Gamma : \Sigma[z] \to \mathbb{S}^2[x]$$

and then exploit the covering map to uplift the calculation of G to the covering surface.

[Lunin-Mathur'00]



Degree of the Covering Map

Number of sheets involved in the covering surface is given by

$$N = 1 + \frac{1}{2}\sum_{j=1}^{n}(w_j - 1)$$

which is the degree of the covering map $\Gamma[z]$

Twist Correlators

On the covering surface the ground state twist fields $\mathcal{O}_{w_i}$ becomes identity $\mathbb{1}$, since their monodromy behaviour is captured by the covering surface itself. So only the conformal factor for the map $\Gamma : \Sigma \to \mathbb{S}^2$ contributes in G:

$$\langle \mathcal{O}_{w_1}(x_1) \cdots \mathcal{O}_{w_n}(x_n) \rangle = \sum_{\Gamma} W_\Gamma \, e^{-S_{\mathrm{L}}[\Phi_\Gamma]} \ ,$$

where

$$S_{\mathrm{L}}[\Phi] = \frac{c}{48\pi} \int d^2 z \sqrt{g} \left(2\, \partial\Phi \, \bar{\partial}\Phi + R\, \Phi \right)$$

with

$$\Phi_\Gamma = \log\, \partial_z \Gamma(z) + \log\, \partial_{\bar{z}} \bar{\Gamma}(\bar{z})$$

.

[Lunin-Mathur'00]

Covering Map requires to solve Matrix Model

Covering map:

$$\partial\Gamma(z) = M_\Gamma \, \frac{\prod_{i=1}^{n-1}(z - z_i)^{w_i - 1}}{\prod_{a=1}^{N}(z - \lambda_a)^2} \ ,$$

The poles are determined by

$$\sum_{i=1}^{n-1} \frac{w_i - 1}{\lambda_a - z_i} = \sum_{b \neq a}^{N} \frac{2}{\lambda_a - \lambda_b} \ , \qquad (a = 1, \ldots, N) \ .$$

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In the large N (i.e large twist) limit, it essentially becomes the saddle-point equation of the Matrix model

$$\mathcal{Z} = \int [dM] e^{-N \, \mathrm{Tr} W(M)} = \int \prod_{a=1}^{N} \frac{d\lambda_a}{2\pi} \prod_{a<b}^{N} (\lambda_a - \lambda_b)^2 \, e^{-N \sum_{a=1}^{N} W(\lambda_a)} \ ,$$

where the potential has a logarithmic Penner-like form

$$W(z) = \sum_{i=1}^{n-1} \alpha_i \log\left(z - z_i\right) \ ,$$

[Gaberdiel-Gopakumar-Knighton-PM'20]

Poles of the Covering Map coalesce in the large N limit

$$\sum_{i=1}^{n-1} \frac{w_i - 1}{\lambda_a - z_i} = \sum_{b \neq a}^{N} \frac{2}{\lambda_a - \lambda_b} \,, \qquad (a = 1, \ldots, N) \,.$$

The dynamics of the eigenvalues $\{\lambda_a\}$ are determined by two forces:
1)Logarithmic attractive potential (l.h.s) and 2)Coulomb repulsion (r.h.s).

In the large N limit, they localize on a set of curves $\mathcal{C}$ on the complex plane.
Spectral curve: [$y(z) = \frac{1}{N}\partial \log \partial \Gamma$]

$$y_0(z) = \frac{\alpha_n}{\prod_{i=1}^{n-1}(z - z_i)} \sqrt{\prod_{k=1}^{2n-4}(z - a_k)} \,.$$

This spectral curve defines a "branching surface" of genus $(n-3)$.

$$2n - 4 = 2\underbrace{(n-3)}_{g} + 2$$

$(n-3)$ A-cylcles and $(n-3)$ B-cycles

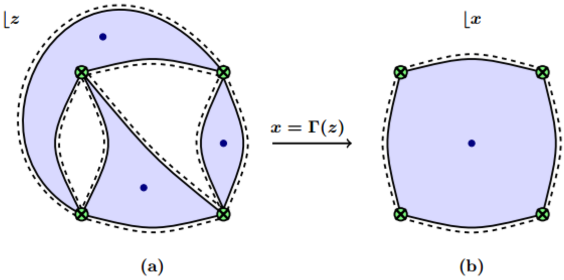
Comments on finite N

$$-\frac{2}{N^2}\,S[\Gamma](z) = y_0^2(z) - \frac{2}{N}W''(z) - \frac{4}{N}R_1(z)$$

$S[\Gamma](z)$: Schwarzian of the covering map

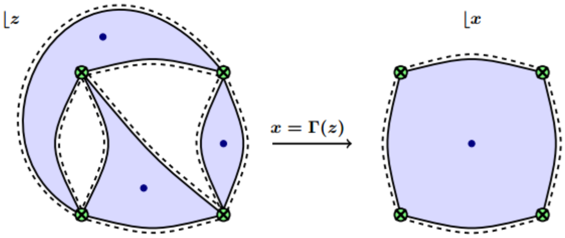
We will observe that $y_0^2(z)$ defines the Strebel metric on the covering surface and it is an open question whether, at finite N, how does the Strebel differential differ from the Schwarzian $S[\Gamma](z)$...

Diagramatics of Symmetric Product CFTs



- "Feynman diagram" for Symmetric product orbifold CFTs is the inverse image (Γ^{-1}) of the Jordan curve (b).

Diagramatics of Symmetric Product CFTs



(a) (b)

- "Feynman diagram" for Symmetric product orbifold CFTs is the inverse image (Γ^{-1}) of the Jordan curve (b).
- Poles of $\Gamma(z)$ $\{\lambda_a\}$ (inverse image of $x = \infty$) are located inside all of the N colored faces.

[Pakman-Rastelli-Razamat'20]

Diagramatics of Symmetric Product CFTs

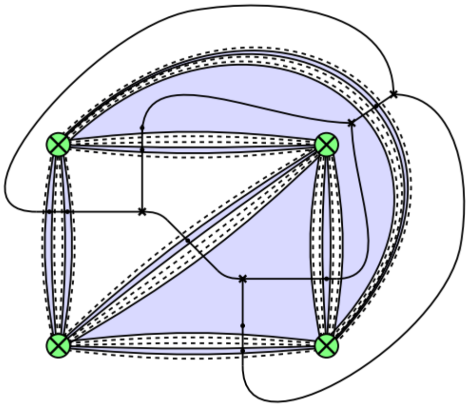
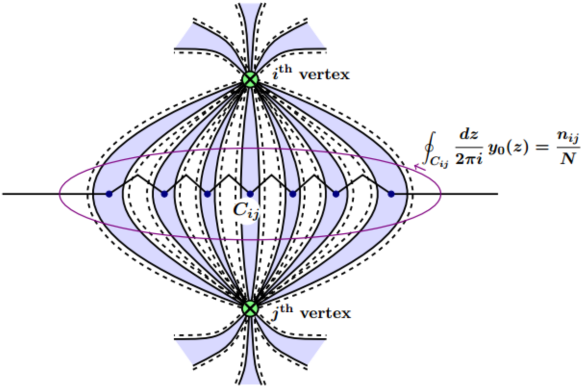


Figure 1: One Feynman graph and it's dual for four-point correlator $\langle \sigma_{[5]}\sigma_{[5]}\sigma_{[5]}\sigma_{[5]} \rangle$

Each diagram defines a distinct covering map.

[Pakman-Rastelli-Razamat'09]

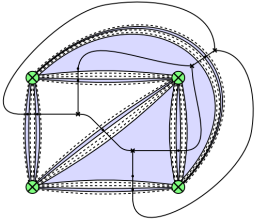
Diagramatics of Symmetric Product CFTs



Poles of the covering map coalesce in the large N to form the edges of the dual skeleton graph.

[Gaberdiel-Gopakumar-Knighton-PM'20]

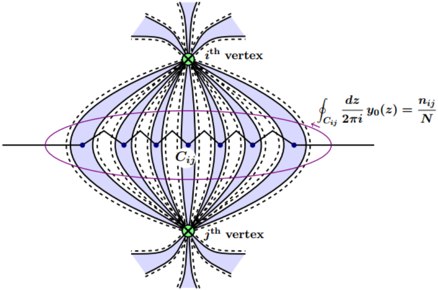
Diagramatics ⇔ Spectral curve



$$y_0(z) = \frac{\alpha_n}{\prod_{i=1}^{n-1}(z - z_i)} \sqrt{\prod_{k=1}^{2n-4}(z - a_k)} \ .$$

The cuts form the dual Feynman graph of the symmetric product orbifold CFT.

Diagramatics $\Leftrightarrow$ Spectral curve



Periods of A and B-cycles $(n_{ij}, \tilde{n}_{ij})$ count the fraction of eigenvalues (Wick contractions) localized in the cuts (dual edge $\widehat{(ij)}$).

$$\frac{1}{4\pi i} \oint_{A_I} y_0(z)dz \equiv \nu_I = \frac{n^{(I)}}{2N}$$

$$\frac{1}{4\pi i} \oint_{B_I} y_0(z)dz \equiv \mu_I = \frac{\tilde{n}^{(I)}}{2N}$$

for $I = 1, \cdots, (n-3)$.

[Gaberdiel-Gopakumar-Knighton-PM'20]

Spectral curve and Strebel differential

The spectral curve defines a special quadratic differential: "Strebel differential"

$$-\frac{1}{4\pi^2}y_0^2(z)dz^2 \equiv \boldsymbol{\phi_S(z)dz^2}.$$

Strebel differential is a quadratic differential holomorphic everywhere except with double poles at n marked points z_i such that all the "lengths" between its zeroes $\{a_k\}$ are real

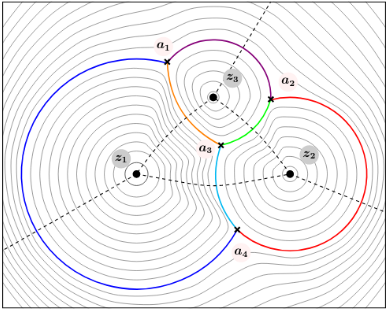
$$l_{km} = \int_{a_k}^{a_m} \sqrt{\phi_S(z)} \in \mathbb{R}_+ \ .$$

The latter condition is clearly satisfied due to

$\frac{1}{4\pi i}\oint_{A_I} y_0(z)dz \equiv \nu_I = \frac{n^{(I)}}{2N}$

$\frac{1}{4\pi i}\oint_{B_I} y_0(z)dz \equiv \mu_I = \frac{\tilde{n}^{(I)}}{2N}$

Strebel graph $\Leftrightarrow$ Dual Feynman graph



Strebel graph $\Leftrightarrow$ Dual Feynman graph

From Fields to Strings

A unique Strebel differential/graph parameterized by $\{l_{ij}\}$ $\xleftrightarrow{\text{Strebel's Theorem}}$ Each point in the decorated Moduli space $(\Sigma_{g,n}|w_1, \cdots, w_n)$

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Spectral curve parameterized by $\{l_A, l_B\}$

$\updownarrow$ [Matrix model analysis]

Each covering map Γ contribution in

$$\langle \mathcal{O}_{[w_1]} \cdots \mathcal{O}_{[w_n]} \rangle = \sum_{\Gamma} e^{-S_L[\Phi_\Gamma]}$$

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$\updownarrow$ [manifestly] $\{l_A, l_B\} = \{l_{ij}\}$

Spectral curve parameterized by $\{l_A, l_B\}$

$\updownarrow$ [Matrix model analysis]

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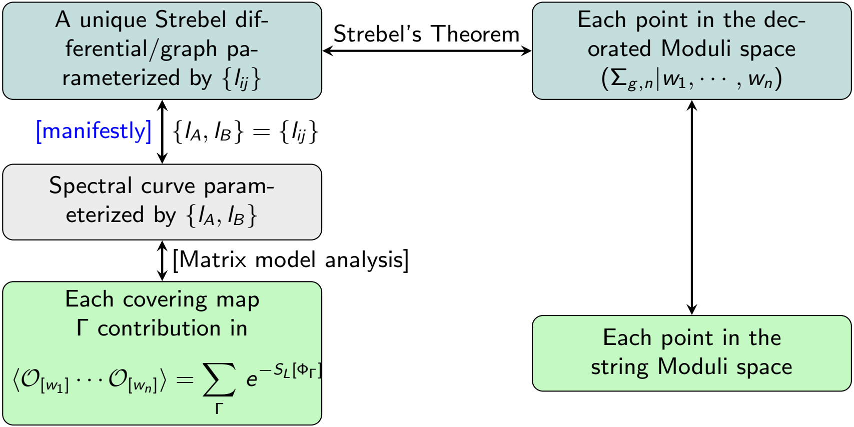
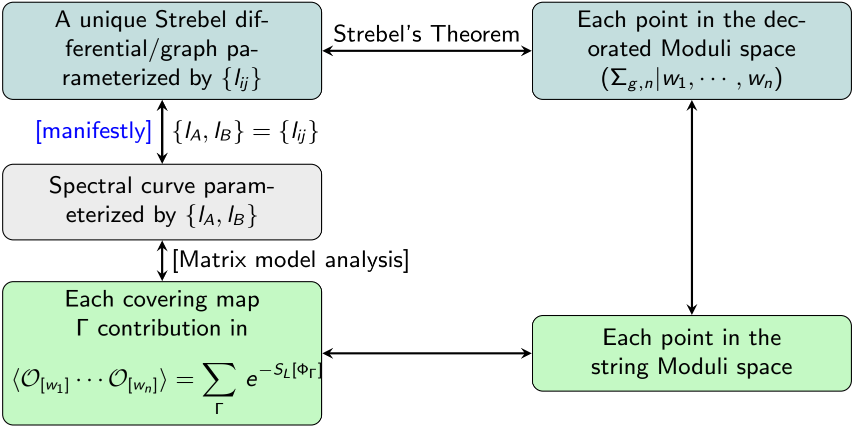
From Fields to Strings

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$\langle \mathcal{O}_{[w_1]} \cdots \mathcal{O}_{[w_n]} \rangle = \sum_{\Gamma} e^{-S_L[\Phi_\Gamma]}$
Each point in the string Moduli space

Reconstructing the Worldsheet

$$\boxed{\langle \mathcal{O}_{w_1}(x_1)\cdots\mathcal{O}_{w_n}(x_n)\rangle = \sum_{\Gamma} W_\Gamma\, e^{-S_{\mathrm{L}}[\Phi_\Gamma]}}$$

Our dictionary:

$$\frac{1}{N}\partial\log\partial\Gamma \;=\; y_0(z) \;=\; i\sqrt{\phi_S(z)}$$

The Liouville action becomes

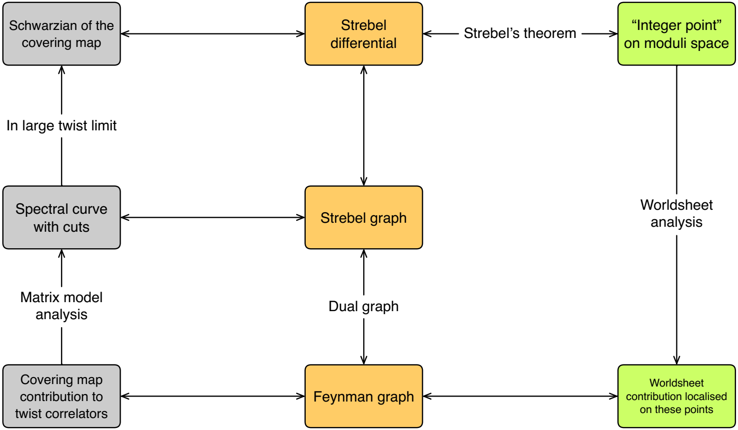
$$\mathrm{S}_L[\Gamma] = \frac{cN^2}{48\pi}\int d^2z\,|\phi_S(z)| = \mathrm{Area}_{\text{Strebel gauge}}\;.$$

which is the Numbu-Goto worldsheet action in "Strebel gauge".

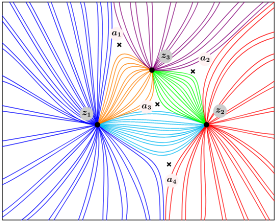
$$\langle \mathcal{O}_{w_1}(x_1)\cdots\mathcal{O}_{w_n}(x_n)\rangle = \sum_{\Gamma} W_\Gamma\, e^{-S_{\mathrm{L}}[\Phi_\Gamma]} \to \int_{\mathcal{M}_{0,n}}[\mathcal{D}m]\,e^{-\mathrm{Area}}$$

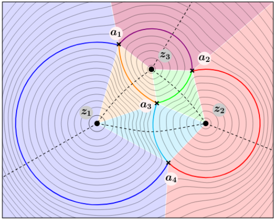
[Gaberdiel-Gopakumar-Knighton-PM'20]

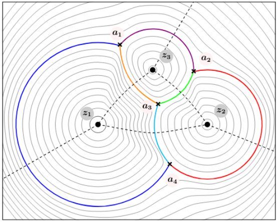
Flowchart


Schwarzian of the covering map
Strebel differential
Strebel's theorem
"Integer point" on moduli space
In large twist limit
Spectral curve with cuts
Strebel graph
Worldsheet analysis
Matrix model analysis
Dual graph
Covering map contribution to twist correlators
Feynman graph
Worldsheet contribution localised on these points

Gluing Graphs into Surfaces







Worldsheet on $AdS_3 \times S^3 \times \mathbb{T}^4$

- Worldsheet theory on $AdS_3 \times S^3 \times \mathbb{T}^4$ with $k = 1$ unit of NS-NS flux is based on supersymmetric WZW model on the group manifold

$$\mathfrak{psu}(1,1|2)_1$$

  together with topologically twisted sigma model for $\mathbb{T}^4$.
- $\mathfrak{psu}(1,1|2)_1$ WZW model has the free field realization in terms of 4 spin-$\frac{1}{2}$ symplectic bosons $\{\xi^{\pm}, \eta^{\pm}\}$ and 4 spin-$\frac{1}{2}$ fermions $\{\psi^{\pm}, \chi^{\pm}\}$.

Twistorial incidence relation

Exploiting the constraints from the OPE of the symplectic bosons $\xi^{\pm}$, $\eta^{\pm}$ with the spectrally flowed vertex operators $V^{w}_{m,j}(x;z)$, the following striking relation holds

$$\langle(\xi^{-}(z)+\Gamma(z)\xi^{+}(z))\rangle_{\text{phys}}=0$$

[Dei-Gaberdiel-Gopakumar-Knighton'20]

where the bracket denote the expectation value with the vertex operators. This suggests a twistorial incidence relation

$$\mu_{\dot{a}}+x_{a\dot{a}}\lambda^{a}=0$$

We can define the Ambi-twistors out of the worlsheet variables

$$Z^{I}=\begin{pmatrix}\xi^{+}\\ \xi^{-}\end{pmatrix},\quad Y_{I}=\begin{pmatrix}-\eta^{-}\\ \eta^{+}\end{pmatrix}$$

where

$$Y_{I}Z^{I}=\xi^{-}\eta^{+}-\xi^{+}\eta^{-}=0,$$

Twistor fields in terms of AdS_3 covering

Wakimoto representation of the $\mathfrak{sl}(2,\mathbb{R})_{k=1}$ currents

$$\boxed{\begin{aligned} J^+ &= \beta \\ J^3 &= -\partial\Phi + \beta\gamma \\ J^- &= -2\gamma\partial\Phi + \beta\gamma\gamma - \partial\gamma \end{aligned}}$$

where $\{\Phi, \gamma, \bar{\gamma}\}$ are identified with the fields parametrizing AdS_3:

$$ds^2_{AdS_3} = d\Phi^2 + e^{2\Phi} d\gamma d\bar{\gamma}$$

[Eberhardt-Gaberdiel-Gopakumar'19]

These currents also have the free field representation in terms of pairs of symplectic bosons $\xi^\pm$ and $\eta^\pm$ of $\mathfrak{psu}(1,1|2)_1$ as

$$\boxed{\begin{aligned} J^3(z) &= -(\eta^+\xi^-)(z) \\ J^\pm(z) &= (\eta^\pm\xi^\pm)(z) \end{aligned}}$$

where

$$\xi^+\eta^- = \xi^-\eta^+$$

Twistor fields in terms of AdS_3 covering

Comparing two representations of the currents

$$
\begin{aligned}
J^{+} &= \beta \\
J^{3} &= -\partial\Phi + \beta\gamma \\
J^{-} &= -2\gamma\partial\Phi + \beta\gamma\gamma - \partial\gamma
\end{aligned}
$$

$$
\begin{aligned}
J^{3} &= -(\eta^{+}\xi^{-}) \\
J^{\pm} &= (\eta^{\pm}\xi^{\pm})
\end{aligned}
$$

with the constraint

$$\boxed{\xi^{+}\eta^{-} = \xi^{-}\eta^{+}}$$

we find the classical twistor solutions

$$\xi^{+} = -\eta^{+} = -\frac{\partial\Phi}{\sqrt{\partial\gamma}}, \quad \xi^{-} = -\eta^{-} = \frac{\gamma\partial\Phi + \partial\gamma}{\sqrt{\partial\gamma}}$$

[Bhat-Gopakumar-PM-Radhakrishnan'21]

Twistor fields in terms of AdS_3 covering

$$\xi^+ = -\eta^+ = -\frac{\partial\Phi}{\sqrt{\partial\gamma}}, \quad \xi^- = -\eta^- = \frac{\gamma\partial\Phi + \partial\gamma}{\sqrt{\partial\gamma}}$$

- We obtain **Stringy** incidence relations

$$\boxed{\begin{aligned} \xi^- + \gamma\xi^+ &= \sqrt{\boldsymbol{\partial\gamma}} \\ \eta^- + \gamma\eta^+ &= -\sqrt{\boldsymbol{\partial\gamma}} \end{aligned}}$$

The right hand side is not zero but rather proportional to the radial profile:

$$r^2(z,\bar{z}) \equiv e^{-2\Phi(z,\bar{z})} = \epsilon^2(\partial\gamma)\,(\bar{\partial}\bar{\gamma})$$

using the classical solution for an n-point correlator:

$$\gamma(z) = \Gamma(z), \quad \Phi(z,\bar{z}) = -\log\epsilon - \frac{1}{2}\log|\partial\Gamma|^2$$

(Sugawara) Stress tensor of $\mathfrak{su}(1,1)_1$ written in terms of free-fields yield

$$T(z) = \frac{1}{2}\left[\left(\frac{\partial^2\gamma}{\partial\gamma}\right) - \frac{1}{2}\left(\frac{\partial^2\gamma}{\partial\gamma}\right)^2\right] = \frac{1}{2}S\left[\gamma(z)\right]$$

where we have used

$$\gamma(z) \equiv \Gamma(z), \quad \partial\Phi(z) = -\frac{1}{2}\partial\log[\partial\Gamma(z)]$$

[Bhat-Gopakumar-PM-Radhakrishnan'21]

Given a covering map

$$\Gamma(z) = \frac{P_N(z)}{Q_N(z)}$$

we can express the twisor variables in terms of the polynomials $P_N(z)$ and $Q_N(z)$:

$$\xi^+ = -\eta^+ = \frac{\tilde{Q}_{N+n-1}(z)}{2\prod_{i=1}^n (z-z_i)^{\frac{(w_i+1)}{2}}} = -\frac{d}{dz}\left[\frac{Q_N(z)}{\prod_{i=1}^n (z-z_i)^{\frac{(w_i-1)}{2}}}\right]$$

and similarly,

$$\xi^- = -\eta^- = -\frac{\tilde{P}_{N+n-1}(z)}{2\prod_{i=1}^n (z-z_i)^{\frac{(w_i+1)}{2}}} = \frac{d}{dz}\left[\frac{P_N(z)}{\prod_{i=1}^n (z-z_i)^{\frac{(w_i-1)}{2}}}\right] .$$

where

$$\tilde{P}_{N+n-1}(z) = \tilde{R}_{n-1}(z)P_N(z) - 2\prod_{i=1}^n (z-z_i)P'_N(z) .$$

and similarly for $\tilde{Q}_{N+n-1}(z)$.

[Bhat-Gopakumar-PM-Radhakrishnan'21]

Worldsheet on $AdS_5 \times S^5$

Recently the worldsheet dual to free 4d $\mathcal{N}=4$ SYM has been proposed to be the free field sigma model with the following field contents:

$$Z^I = (\lambda^\alpha, \mu^{\dot\alpha}, \psi^a) = (\lambda^1, \lambda^2, \mu^1, \mu^2, \psi^1, \psi^2, \psi^3, \psi^4)$$
$$Y_J = (\mu^\dagger_\beta, \lambda^\dagger_{\dot\beta}, \psi^\dagger_a) = (\mu^\dagger_1, \mu^\dagger_2, \lambda^\dagger_1, \lambda^\dagger_2, \psi^\dagger_1, \psi^\dagger_2, \psi^\dagger_3, \psi^\dagger_4)$$

while they obey the "ambi-twistor" constraint

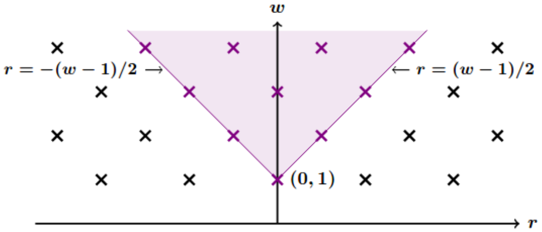
$$Y_I Z^I = 0$$

This worldsheet model precisely reproduces the spectrum of free 4d $\mathcal{N}=4$ SYM.

[Gaberdiel-Gopakumar'21]

New perspective: Sigma model on the **twistorial** $AdS_5 \times S^5$ target space

Wedge modes



'Physical' gauge constraints:

Only modes $(Z^I)_r, (Y_J)_r$ lying within the wedge $-\frac{w-1}{2} \leq r \leq \frac{w-1}{2}$ are excited

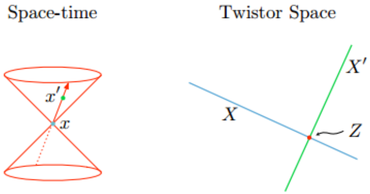
We can view these "wedge modes" as exciting a discrete set of w string bits localised along the worldsheet.

We want to discuss a classical analysis of AdS_5 twistor fields and see how the notion of covering maps generalize to this case.

Twistor space of $\mathbb{M}_{\mathbb{C}}$

Twistor space of the complexified Minkowski space $\mathbb{M}_{\mathbb{C}}$ corresponds to open subset of $\mathbb{CP}^3$ with homogeneous co-ordinates

$$Z^I = (\lambda^\alpha, \mu^{\dot\alpha}), \quad \alpha, \dot\alpha = 1, 2$$



where a point $x^{\alpha\dot\alpha} \in \mathbb{M}_{\mathbb{C}}$ corresponds to a complex "line" in twistor space

$$\mu^{\dot\alpha} = x^{\alpha\dot\alpha}\,\lambda_\alpha$$

which can be rephrased as

$$\underbrace{\begin{bmatrix} \frac{1}{2}x^2\epsilon_{\alpha\beta} & -x_{\alpha\dot\beta} \\ x_{\dot\alpha\beta} & \epsilon_{\dot\alpha\dot\beta} \end{bmatrix}}_{X_{IJ}} \underbrace{\begin{bmatrix} \lambda^\alpha \\ \mu^{\dot\alpha} \end{bmatrix}}_{Z^J} = 0$$

AdS_5 from Projective space

X^{IJ}: skewsymmetric 4×4 matrix with projective invariance $X \to \lambda X$, parameterizing $\mathbb{CP}^5$.

Then the metric

$$ds^2 = -\frac{dX^2}{X^2} + \left(\frac{X \cdot d\,X}{X^2}\right)^2$$

with

$$X^{IJ} = (X_b)^{IJ} + \frac{r^2}{2} I^{IJ}$$

becomes the AdS_5 metric (in Poincare co-ordinates),

$$ds^2 = \frac{dr^2 + dx_{\alpha\dot\beta}dx^{\alpha\dot\beta}}{r^2}$$

1. $$X_b^{IJ} = \begin{bmatrix} \epsilon^{\alpha\beta} & x^{\alpha\dot\beta} \\ -x^{\dot\alpha\beta} & \frac{1}{2}x^2\epsilon^{\dot\alpha\dot\beta} \end{bmatrix}$$ parameterizes the boundary of AdS_5. $[(X_b^{IJ})^2 = 0]$

2. $$I^{IJ} = \begin{bmatrix} 0 & 0 \\ 0 & \epsilon^{\dot\alpha\dot\beta} \end{bmatrix}$$

Defining (Ambi-) Twistors

We now define the twistors (open subsets of $\mathbb{CP}^3$)

$$Z^I = (\lambda^\alpha, \mu^{\dot\alpha}) = (\lambda^1, \lambda^2, \mu^1, \mu^2)$$
$$Y_J = (\mu^\dagger_\beta, \lambda^\dagger_{\dot\beta}) = (\mu^\dagger_1, \mu^\dagger_2, \lambda^\dagger_1, \lambda^\dagger_2)$$

These will play the role of ambitwistors for ∂AdS_5, but more generally twistor variables for the bulk AdS_5.

Incidence relation on the boundary:

$$(X_b)_{IJ}\, Z_b^J = 0 \iff \mu^{\dot\alpha} = x^{\dot\alpha}{}_\beta \lambda^\beta$$
$$X_b^{IJ}\, Y_J^b = 0 \iff \mu^\dagger_\alpha = -x_\alpha{}^{\dot\beta} \lambda^\dagger_{\dot\beta}\ .$$

These imply

$$\mathcal{C}_b \equiv Z_b^I Y_I^b = 0$$

.

Twistor space of AdS_5

Incidence relation in the bulk:

$$Z^I = X^{IJ} Y_J$$
$$\Rightarrow \begin{bmatrix} \lambda^{\alpha} \\ \mu^{\dot\alpha} \end{bmatrix} = \begin{bmatrix} \epsilon^{\alpha\beta} & x^{\alpha\dot\beta} \\ -x^{\dot\alpha\beta} & \frac{1}{2}(x^2 + r^2)\epsilon^{\dot\alpha\dot\beta} \end{bmatrix} \begin{bmatrix} \mu^{\dagger}_{\beta} \\ \lambda^{\dagger}_{\dot\beta} \end{bmatrix} .$$

This automatically satisfies the ambitwistor constraint:

$$\boxed{\mathcal{C} = Z^I Y_I = 0}$$

which arises a fundamental gauge constraint [making quotient in $\mathfrak{psu}(2,2|4)_1$] from the worldsheet analysis.

[Adamo-Skinner-Williams'16]

As $r \to 0$, the bulk AdS_5 incidence relations reduces to those on the boundary.

Twistor covering maps in AdS_5

To describe the string configurations which capture the dual $\mathcal{N} = 4$ SYM, we promote the twistor variables and the AdS_5 spacetime as fields on the worldsheet:

$$X^{IJ}(z, \bar{z}),\ Z^I(z),\ Y_J(z),\ \widehat{Z}^I(\bar{z}),\ \widehat{Y}_J(\bar{z})$$

The AdS_5 incidence relations hold point-wise

$$\boxed{Z^I(z) = X^{IJ}(z, \bar{z}) Y_J(z)}$$

Twistor covering maps in AdS_5

For string configuration near the boundary of AdS_5

$$\mu^{\dot\alpha}(z) = X^{\dot\alpha}{}_{\beta}(z,\bar z)\lambda^{\beta}(z)$$
$$\mu^{\dagger}_{\alpha}(z) = -X_{\alpha}{}^{\dot\beta}(z,\bar z)\lambda^{\dagger}_{\dot\beta}(z)\ .$$

Clearly then

$$\bar\partial X^{\dot\alpha}{}_{\beta}(z,\bar z)\lambda^{\beta}(z) = 0\ ,$$
$$\bar\partial X_{\alpha}{}^{\dot\beta}(z,\bar z)\lambda^{\dagger}_{\dot\beta}(z) = 0\ .$$

so that

$$\bar\partial X^{\dot\alpha}{}_{\beta}(z,\bar z) = \begin{bmatrix} 0 & 0 \\ 0 & -\bar\partial\bar V(\bar z)\end{bmatrix} \implies \quad X^{\dot\alpha}{}_{\beta}(z,\bar z) = \begin{bmatrix} -V(z) & 0 \\ 0 & -\bar V(\bar z)\end{bmatrix}\ .$$

Locally we can always view $X^{\dot\alpha}{}_{\beta}(z,\bar z)$ as a holomorphic embedding into the boundary of an AdS_3 subspace of the bulk AdS_5 spacetime.

Maps to an AdS_3 subspace

We restrict to the kinematic set up where the boundary insertion points $\{x_i\}$ lie within a two-dimensional plane. (Not a restriction for 2,3 and 4-point functions of $\mathcal{N}=4$ SYM), so that (globally) string lies on the boundary of the AdS_3 subspace.

$$X^{\dot\alpha}{}_{\beta}(z,\bar z) = \begin{bmatrix} -V(z) & 0 \\ 0 & -\bar V(\bar z) \end{bmatrix},\ \lambda^{\beta}(z) = \begin{bmatrix} \lambda^1(z) \\ 0 \end{bmatrix} \Longrightarrow \mu^{\dot\alpha}(z) = -\begin{bmatrix} V(z)\lambda^1(z) \\ 0 \end{bmatrix}.$$

$V(z) = -\mu^1(z)/\lambda^1(z)$ is a covering map from the genus zero worldsheet to the S^2 boundary of the AdS_3. Generalizing the results for AdS_3 to this set up:

$$\lambda^1(z) = \frac{R_{n-1}(z)Q^1_N(z)}{\prod_{i=1}^n (z-z_i)^{\frac{w_i}{2}}}\ ,\ \ \mu^1(z) = \frac{R_{n-1}(z)P^1_N(z)}{\prod_{i=1}^n (z-z_i)^{\frac{w_i}{2}}}\ .$$

This ties up with $Z^I(z) = \sum_{r=-\frac{w_i-1}{2}}^{\frac{w_i-1}{2}} \frac{(Z^I)_r}{(z-z_i)^{r+1/2}}$

[Bhat-Gopakumar-PM-Radhakrishnan'21]

Feynman covering

Correlator of n gauge-invariant scalar operators in the free $\mathcal{N} = 4$ SYM:

$$\langle \mathcal{O}^{(w_1)}(x_1) \cdots \mathcal{O}^{(w_n)}(x_n) \rangle = \sum_{\{n_{ij}\}} C_{\{n_{ij}\}} \prod_{(i,j)} \left(\frac{1}{x_{ij}^2} \right)^{n_{ij}}$$

For the two-point function joining (x_i, x_j): $\left(\frac{1}{x_{ij}^2} \right)^{w}$, consider the following covering map

$$\Gamma(z) = \frac{V_j \, z^w + V_i}{z^w + 1} = \frac{P_w(z)}{Q_w(z)}$$

$$\text{with } V_k = x_k^{(1)} + i x_k^{(2)} \; .$$

Note that two points (x_i, x_j) can always be taken to lie on a plane, corresponding states on the worldsheet are inserted at $z = 0$ and $z = \infty$.

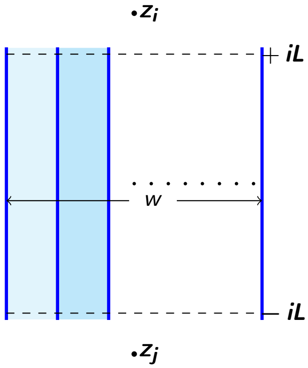
[Bhat-Gopakumar-PM-Radhakrishnan'21]

Feynman covering

It is convenient to view the covering map in u coordinate, where

$$z = \exp\left[2\pi i \frac{u}{w}\right]$$

mapping a vertical strip ($0 < \mathrm{Re}\, u \leq w$) onto the sphere such that $z = (0, \infty)$ are images of $u = \pm i\infty$, respectively, on the strip.



$$\boxed{\Gamma(u) \equiv V(z(u)) = \frac{V_i + V_j}{2} + \frac{V_i - V_j}{2i}\, \tan(\pi u)}$$

Feynman covering

$$\boxed{\Gamma(u) \equiv V(z(u)) = \frac{V_i + V_j}{2} + \frac{V_i - V_j}{2i}\ \tan(\pi u)}$$

This is essentially the unique map for which the Schwarzian is a constant:

$$S[\Gamma(u)] = \frac{\Gamma'''}{\Gamma'} - \frac{3}{2}\left(\frac{\Gamma''}{\Gamma'}\right)^2 = 2\pi^2\ .$$

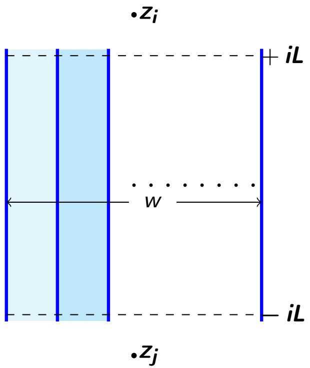
The unique Strebel quadratic differential on the strip with poles only at $u = \pm i\infty$ is also just du^2. Thus

$$\phi_S(u)\, du^2 = \frac{1}{2\pi^2} S[\Gamma(u)]\, du^2\ .$$

This is a coordinate independent statement.

Feynman covering

The strip of width w is nothing other than the w double line edges glued together



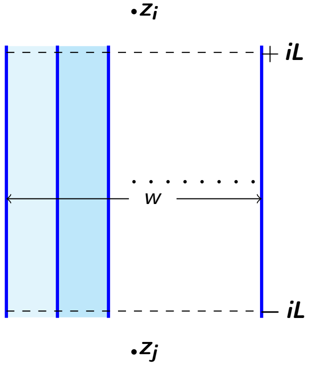
We can try to compute the Nambu-Goto area of the worldsheet in the "Strebel gauge":

$$ds^2 = |\phi_S(z)|\, dz d\bar{z}$$

Strebel area of the covering surface

In the u-coordinate, the textcolorviolet"Strebel" area of the strip becomes $A_{ij} = 2Lw$, where we've introduced cutoff in spacetime ϵ:

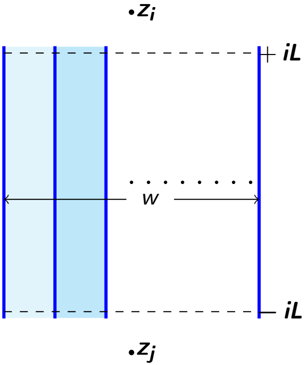
$$|V_i - \Gamma(u = iL)| = |V_j - \Gamma(u = -iL)| = \epsilon$$



We find,

$$\boxed{L = \frac{1}{4\pi^2} \log\left(\frac{x_{ij}^2}{\epsilon^2}\right)}$$

Covering surface area computes Feynman propagator

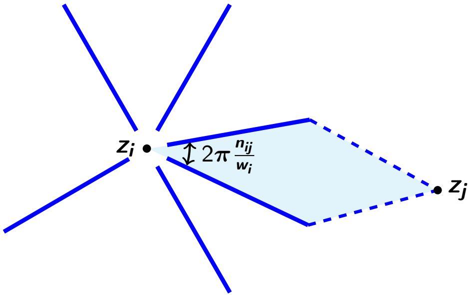


The Nambu-Goto weight with the "Strebel" area of the strip is then

$$\exp[-2\pi A_{ij}] = \exp[2\pi \times 2Lw] = \epsilon^{2w} \left(\frac{1}{x_{ij}^2} \right)^{w}.$$

[Bhat-Gopakumar-PM-Radhakrishnan'21]

For multi-point correlator



Adding the areas of the strips near a vertex of the n-point correlator,

$$\langle \mathcal{O}_{w_1}(x_1) \cdots \mathcal{O}_{w_n}(x_n) \rangle = \sum_{\{n_{ij}\}} C_{\{n_{ij}\}} \exp \left[2\pi \sum_{i<j} A_{ij} \right]$$

[Bhat-Gopakumar-PM-Radhakrishnan'21]

Lessons (*take home message*)

1. Each Feynman graph can be associated to a point in the closed string moduli space, via the Strebel correspondence. Sum over all feynman diagrams defining the correlator goes over (in large twist limit) to an integral over this moduli space.

2. Worldsheet fields can be seen as holomorphic covering map to the twistorial target space.

Outlook

1. To obtain the AdS_5 twistor incidence relation from worldsheet analysis.

   [Gaberdiel-Gopakumar-PM-Knighton, in progress]

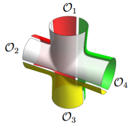
2. How does the Strebel differential deform away from the Schwarzian of covering map if we take $1/N$ corrections?

   [Gopakumar-PM-Sarkar, in progress]

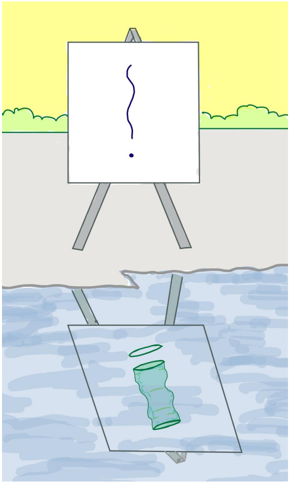
3. Towards the string dual of 2d Yang Mills ...

   [Komatsu-PM, in progress]

4. Making connection with the Hexagonalization program of $\mathcal{N} = 4$ SYM.

$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle =$ 

5. Making connection with the Mellin amplitudes of the perturbative $\mathcal{N} = 4$ SYM.



Thanks for your attention