

From Fields to Strings

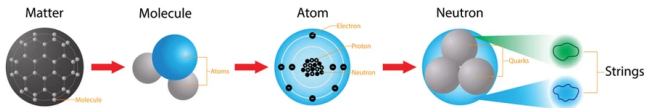
Pronobesh Maity

Based on (Gaberdiel-Gopakumar-Knighton-PM - [arXiv:2011.10038](#)) and
(Bhat-Gopakumar-PM-Radhakrishnan - [arXiv:2112.05115](#))

September 15, 2022

Dominant viewpoint in the pre-AdS/CFT era: **Strings** as the ultimate building blocks of nature

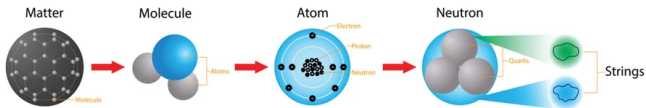
“Reductionism”



[Image taken from Google]

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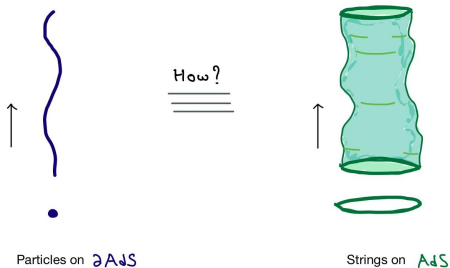
[Image taken from Google]

A different perspective after AdS/CFT: **Strings** as emergent geometry from **Gauge theories** (at large N).

“Emergence”

Motivation

How exactly do quantum field theories organise themselves into string theories in the large N limit? [t Hooft '74]



Maldacena's $AdS=CFT$ concretizes such dualities, but both sides (boundary and bulk) are not decipherable simultaneously in the proposed examples, making it hard to portray the **mechanism** behind it. [\[Maldacena'97\]](#)

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Recent progress on $AdS_3=CFT_2$ and $AdS_5=CFT_4$ overcomes this difficulty giving explicit description of tensionless strings in the bulk, dual to free CFTs on the boundary, in the large N limit.
[\[Eberhardt-Gaberdiel-Gopakumar'18-'19\]](#) [\[Gaberdiel-Gopakumar'21\]](#)

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Gopakumar's program of gluing worldlines into worldsheet **proposes** one **mechanism** for Gauge/String dualities. [Gopakumar'03-05]

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Gopakumar's program of gluing worldlines into worldsheet **proposes** one **mechanism** for Gauge/String dualities. [Gopakumar'03-05]

Can we revisit Gopakumar's construction in the context of these exact dualities?

Motivation: Output

We will show how Gopakumar's prescription works **explicitly** for $AdS_3=CFT_2$ and (*to some extent*) for $AdS_5=CFT_4$:

Feynman Graph , Strebel Graph , Point in the String moduli space.

Proposal \Rightarrow **Explicit** Construction

1. Symmetric Product CFTs and **Covering surfaces**, and **Matrix model**,
2. **Feynman graph** , **Matrix model solution** , Strebel graph , point in the String moduli space,
3. **Twistor covering** in AdS_3 ,
4. **Twistor covering** in AdS_5 ,
5. An explicit **covering map** for AdS_5 whose area reproduces **Feynman propagator**.

Symmetric Product Orbifold CFT

We will consider the symmetric product orbifold CFT

$$\text{Sym}^K(\mathcal{M}) = (\mathcal{M})^K / S_K$$

which corresponds to taking K copies of sigma models on \mathcal{M} and identifying them by the permutation group S_K .

Untwisted sector: states invariant under the permutation.

Twisted sector: Single-cycle twist field ϕ_w (w is the cycle length)

$$\phi_w(z) O_w(\sigma) = \phi_w(\sigma(z)) O_w(\sigma)$$

Twist operator O_w induces a w -fold cyclic permutation amongst some of the K copies of the seed theory.

Twist Correlators and Covering Map

$$G = \langle h O_{w_1}(x_1) \dots O_{w_n}(x_n) \rangle$$

Locally the w -cycle twist field $O_w(x_0)$ induces a w -fold covering

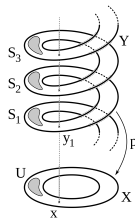
$$z \mapsto \Gamma(z) = x_0 + a(z - z_0)^w + \dots$$

We can combine these local coverings near all the insertion points into an (auxiliary) global covering surface Σ described by the covering map

$$\Gamma : \Sigma[z] \rightarrow S^2[x]$$

and then exploit the covering map to uplift the calculation of G to the covering surface.

[Lunin-Mathur'00]



Degree of the Covering Map

Number of sheets involved in the covering surface is given by

$$N = 1 + \frac{1}{2} \sum_{j=1}^n (w_j - 1)$$

which is the **degree** of the covering map $\Gamma[z]$

On the **covering surface** the ground state twist fields O_{w_i} becomes identity 1, since **their monodromy behaviour is captured by the covering surface itself**. So only the conformal factor for the map $\Gamma : \Sigma \rightarrow S^2$ contributes in G :

$$\langle h O_{w_1}(x_1) \dots O_{w_n}(x_n) \rangle = \int_{\Gamma} W_{\Gamma} e^{-S_L[\Phi_{\Gamma}]} ;$$

where

$$S_L[\Phi] = \frac{c}{48} \int d^2z \sqrt{g} \left(2 \partial\Phi \bar{\partial}\Phi + R\Phi \right)$$

with

$$\Phi_{\Gamma} = \log \partial_z \Gamma(z) + \log \bar{\partial}_{\bar{z}} \bar{\Gamma}(\bar{z})$$

Covering Map requires to solve Matrix Model

Covering map:

$$\omega(z) = M \prod_{i=1}^n \frac{(z - z_i)^{w_i - 1}}{(z - a)^2} ;$$

The poles are determined by

$$\sum_{i=1}^n \frac{w_i - 1}{z_i} = \sum_{b \in a} \frac{2}{a - b} ; \quad (a = 1; \dots; N) :$$

Covering Map requires to solve Matrix Model

Covering map:

$$\mathcal{Z} \Gamma(z) = M \int \prod_{i=1}^N \frac{(z - z_i)^{w_i - 1}}{(z - a)^2} dz ;$$

The poles are determined by

$$\sum_{i=1}^N \frac{w_i - 1}{z - z_i} = \sum_{b \neq a} \frac{2}{z - a} ; \quad (a = 1; \dots; N) ;$$

In the large N (i.e large twist) limit, it essentially becomes the **saddle-point equation** of the Matrix model

$$Z = \int [dM] e^{N \text{Tr} W(M)} = \int \prod_{a=1}^N \frac{d^2 a}{2} \prod_{a < b} (a - b)^2 e^{-N \sum_{a=1}^N W(a)} ;$$

where the potential has a logarithmic Penner-like form

$$W(z) = \sum_{i=1}^N \log(z - z_i) ;$$

Poles of the Covering Map coalesce in the large N limit

$$\sum_{i=1}^N \frac{w_i}{a} \frac{1}{z_i} = \sum_{b \neq a}^N \frac{2}{a} \frac{1}{b} ; \quad (a = 1; \dots; N) :$$

The dynamics of the eigenvalues $f_a g$ are determined by two forces:
 1) Logarithmic attractive potential (l.h.s) and 2) Coulomb repulsion (r.h.s).

In the large N limit, they localize on a set of curves \mathcal{C} on the complex plane.
 Spectral curve: $[y(z) = \frac{1}{N} \log \Gamma]$

$$y_0(z) = \Theta \frac{n}{\prod_{i=1}^n (z - z_i)} \prod_{k=1}^g \frac{1}{(z - a_k)^{2g_k}} :$$

This spectral curve defines a “branching surface” of genus $(n - 3)$.

$$2n - 4 = 2 \left(\underbrace{n - 3}_g \right) + 2$$

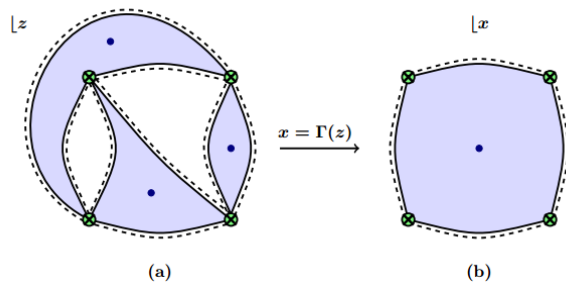
$(n - 3)$ A-cycles and $(n - 3)$ B-cycles

$$\frac{2}{N^2} S[\Gamma](z) = y_0^2(z) - \frac{2}{N} W^{00}(z) - \frac{4}{N} R_1(z)$$

$S[\Gamma](z)$: Schwarzian of the covering map

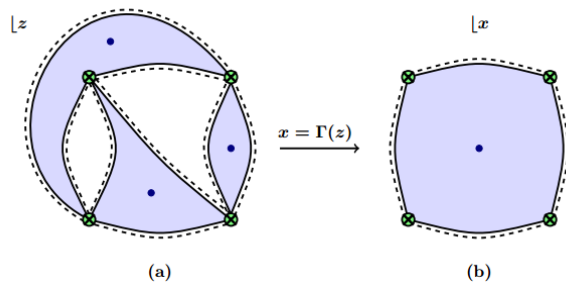
We will observe that $y_0^2(z)$ defines the **Strebel metric** on the covering surface and it is an **open question** whether, at finite N , how does the **Strebel differential** differ from the **Schwarzian** $S[\Gamma](z)$...

Diagrammatics of Symmetric Product CFTs



- | “Feynman diagram” for Symmetric product orbifold CFTs is the inverse image (Γ^{-1}) of the Jordan curve (b).

Diagrammatics of Symmetric Product CFTs



- | “Feynman diagram” for Symmetric product orbifold CFTs is the inverse image (Γ^{-1}) of the Jordan curve (b).
- | Poles of $\Gamma(z)$ (inverse image of $x = 1$) are located inside all of the N colored faces.

[Pakman-Rastelli-Razamat'20]

Diagrammatics of Symmetric Product CFTs

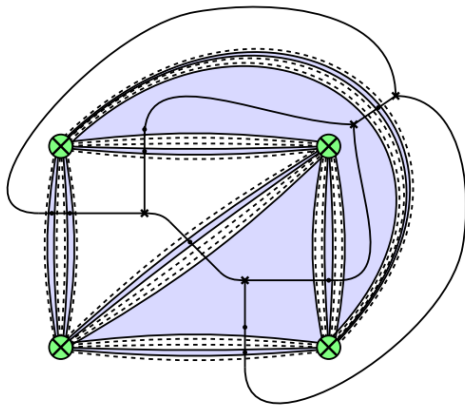


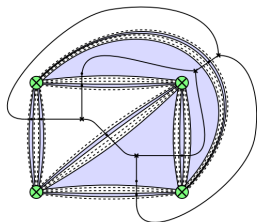
Figure 1: One Feynman graph and its dual for four-point correlator $\langle \sigma_{[5]} \sigma_{[5]} \sigma_{[5]} \sigma_{[5]} \rangle$

Each diagram defines a distinct covering map.

[Pakman-Rastelli-Razamat'09]

Poles of the covering map coalesce in the large N to form the edges of the dual skeleton graph.

[Gaberdiel-Gopakumar-Knighton-PM'20]



$$y_0(z) = \Theta_{n-1}^n(z, z_i) \prod_{k=1}^n \frac{1}{z - a_k} :$$

The cuts form the dual Feynman graph of the symmetric product orbifold CFT.

Periods of A and B-cycles $(n_{ij}; \mathfrak{n}_{ij})$ count the fraction of eigenvalues (**Wick contractions**) localized in the cuts (**dual edge e_{ij}**).

$$\frac{1}{4} \int_{A_l} \text{H} y_0(z) dz \quad l = \frac{n^{(l)}}{2N}$$

$$\frac{1}{4} \int_{B_l} \text{H} y_0(z) dz \quad l = \frac{\mathfrak{n}^{(l)}}{2N}$$

for $l = 1; \dots; (n-3)$.

Spectral curve and Strebel differential

The spectral curve defines a special quadratic differential: "Strebel differential"

$$\frac{1}{4} y_0^2(z) dz^2 = s(z) dz^2:$$

"Strebel differential" is a quadratic differential holomorphic everywhere except with double poles at marked points z_i such that all the "lengths" between its zeroes a_k are real

$$l_{km} = \int_{a_k}^{a_m} \sqrt{s(z)} dz \in \mathbb{R}_+ :$$

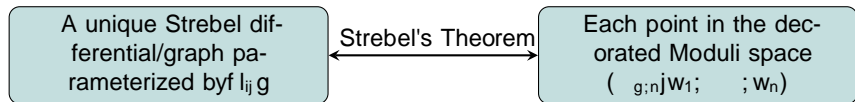
The latter condition is clearly satisfied due to

$$\frac{1}{4} \int_{A_j}^H y_0(z) dz \quad l = \frac{n^{(j)}}{2N}$$

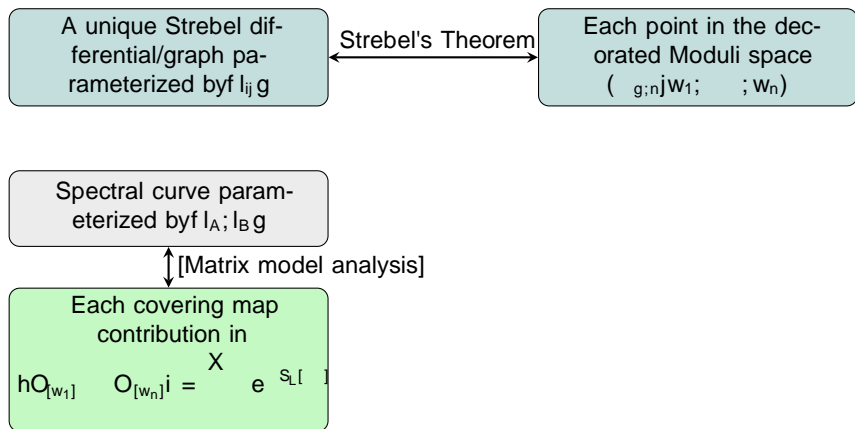
$$\frac{1}{4} \int_{B_l}^H y_0(z) dz \quad l = \frac{n^{(l)}}{2N}$$

Strebel graph, Dual Feynman graph

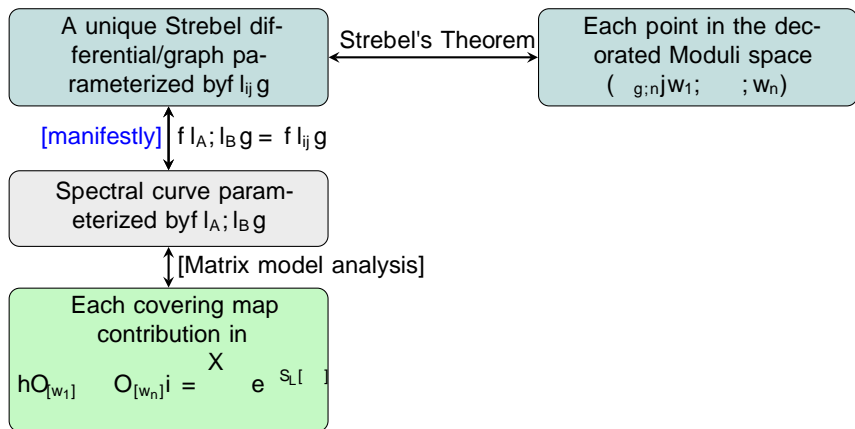
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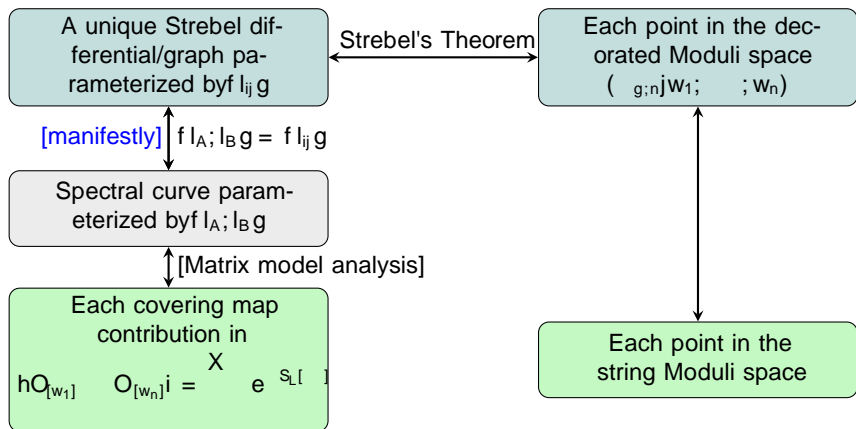
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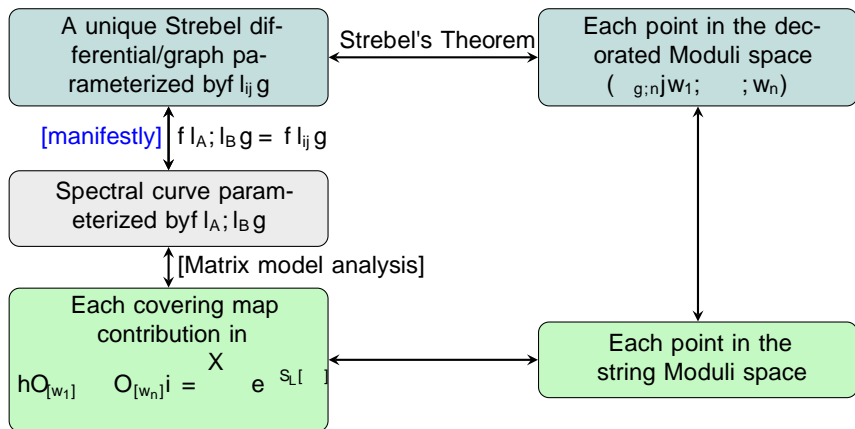
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From Fields to Strings



From Fields to Strings



Reconstructing the Worksheet

$$\langle h_{\mathcal{O}_{w_1}}(x_1) \dots \mathcal{O}_{w_n}(x_n) \rangle = \int \mathcal{W} e^{\mathcal{S}_L[\gamma]}$$

Our dictionary:

$$\frac{1}{N} \text{tr} \log @ = y_0(z) = i^p \overline{s(z)}$$

The Liouville action becomes

$$\mathcal{S}_L[\gamma] = \frac{cN^2}{48} \int d^2z j_s(z) j = \text{Area}_{\text{Strebel gauge}} :$$

which is the Nambu-Goto worldsheet action in "Strebel gauge".

$$\langle h_{\mathcal{O}_{w_1}}(x_1) \dots \mathcal{O}_{w_n}(x_n) \rangle = \int \mathcal{W} e^{\mathcal{S}_L[\gamma]} \int_{M_{0,n}} [Dm] e^{\text{Area}}$$

Flowchart

Gluing Graphs into Surfaces

- | Worldsheet theory on $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ with $k = 1$ unit of NS-NS flux is based on supersymmetric WZW model on the group manifold

$$\text{psu}(1; 1|2)_1$$

together with topologically twisted sigma model for T^4 .

- | $\text{psu}(1; 1|2)_1$ WZW model has the free field realization in terms of 4 spin^1_2 -symplectic bosons $f^{\pm, \pm}$; g and 4 spin^1_2 -fermions $f^{\pm, \pm}$; g .

Twistorial incidence relation

Exploiting the constraints from the OPE of the symplectic bosons, with the spectrally flowed vertex operators $V_{m;j}^w(x; z)$, the following striking relation holds

$$\langle \psi(z) + \psi^\dagger(z) \rangle_{\text{phys}} = 0$$

[Dei-Gaberdiel-Gopakumar-Knighton'20]

where the bracket denote the expectation value with the vertex operators. This suggests a twistorial incidence relation

$$\bar{x}_a + x_{aa}^a = 0$$

We can define the [Ambi-twistors](#) out of the worksheet variables

$$Z^I = \psi^\dagger; \quad Y_I = \psi +$$

where

$$Y_I Z^I = \psi^\dagger + \psi = 0;$$

Twistor fields in terms of AdS_3 covering

Wakimoto representation of the $sl(2; \mathbb{R})_{k=1}$ currents

$$\begin{aligned} J^+ &= \\ J^3 &= \partial_+ \\ J^- &= 2\partial_+ + \partial_- \end{aligned}$$

where ∂_\pm ; g are identified with the fields parametrizing AdS_3 :

$$ds_{AdS_3}^2 = dt^2 + e^{2t} dx^2$$

[Eberhardt-Gaberdiel-Gopakumar'19]

These currents also have the free field representation in terms of pairs of symplectic bosons α and β of $psu(1; 1|2)_1$ as

$$\begin{aligned} J^3(z) &= (\alpha^+ \beta^+)(z) \\ J^-(z) &= (\alpha^- \beta^-)(z) \end{aligned}$$

where

$$\alpha^+ = \alpha + \beta$$

Twistor fields in terms of AdS_3 covering

Comparing two representations of the currents

$$J^+ =$$

$$J^3 = @ +$$

$$J = 2 @ + @$$

$$J^3 = (+)$$

$$J = ()$$

with the constraint

$$+ = +$$

we find the classical twistor solutions

$$+ = + = p \frac{@}{@}; \quad = = \frac{@ + @}{p @}$$

Twistor fields in terms of AdS₃ covering

$$+ = + = p \frac{+}{+}; \quad = = \frac{+ +}{+}$$

- We obtain **Stringy incidence relations**

$$\begin{aligned} + + &= p \frac{+}{+} \\ + + &= p \frac{+}{+} \end{aligned}$$

The right hand side is not zero but rather proportional to the radial profile:

$$r^2(z; z) e^{2(z; z)} = {}^2(+)(+)$$

using the classical solution for am-point correlator:

$$(z) = (z); \quad (z; z) = \log \frac{1}{2} \log j @ j^2$$

(Sugawara) Stress tensor $T(z)$ written in terms of free-elds yield

$$T(z) = \frac{1}{2} \left(\frac{\partial \phi}{\partial z} \right)^2 = \frac{1}{2} S[\phi(z)]$$

where we have used

$$\phi(z) = \phi(z); \quad \partial \phi(z) = \frac{1}{2} \partial \log[\partial \phi(z)]$$

[Bhat-Gopakumar-PM-Radhakrishnan'21]

Given a covering map

$$(z) = \frac{P_N(z)}{Q_N(z)}$$

we can express the twistor variables in terms of the polynomials $P_N(z)$ and $Q_N(z)$:

$$+ = + = \frac{Q_{N+n-1}(z)}{2 \prod_{i=1}^n (z - z_i)^{\frac{(w_i+1)}{2}}} = \frac{d^h}{dz} \frac{Q_N(z)}{\prod_{i=1}^n (z - z_i)^{\frac{(w_i-1)}{2}}}$$

and similarly,

$$= = \frac{P_{N+n-1}(z)}{2 \prod_{i=1}^n (z - z_i)^{\frac{(w_i+1)}{2}}} = \frac{d^h}{dz} \frac{P_N(z)}{\prod_{i=1}^n (z - z_i)^{\frac{(w_i-1)}{2}}} :$$

where

$$P_{N+n-1}(z) = \prod_{i=1}^n (z - z_i) P_N(z)$$

and similarly for $Q_{N+n-1}(z)$.

Recently the worldsheet dual to free $AdS_5 \times S^5$ SYM has been proposed to be the free world sigma model with the following world contents:

$$Z^I = (z^1; z^2; a) = (z^1; z^2; 1; z^2; 1; z^2; 3; 4)$$

$$Y_J = (y^1; y^2; a) = (y^1; y^2; 1; y^2; 1; y^2; 2; 3; 4)$$

while they obey the "ambitwistor" constraint

$$Y_I Z^I = 0$$

This worldsheet model precisely reproduces the spectrum of free $AdS_5 \times S^5$ SYM.

[Gaberdiel-Gopakumar'21]

New perspective: Sigma model on the twistorial $AdS_5 \times S^5$ target space

Wedge modes

'Physical' gauge constraints:

Only modes $(Z^I)_r; (Y_J)_r$ lying within the wedge $\frac{w-1}{2} \leq r \leq \frac{w+1}{2}$ are excited

We can view these "wedge modes" as exciting a discrete set of string bits localised along the worldsheet.

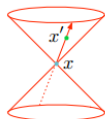
We want to discuss a **classical analysis** of AdS_5 twistor fields and see how the notion of **covering maps** generalize to this case.

Twistor space of M_C

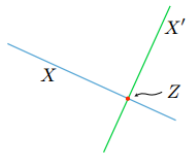
Twistor space of the complexified Minkowski space M_C corresponds to open subset of CP^3 with homogeneous co-ordinates

$$Z^I = (\quad ; \quad); \quad ; \quad = 1;2$$

Space-time



Twistor Space



where a point $x \in M_C$ corresponds to a complex "line" in twistor space

$$= x$$

which can be rephrased as

$$\frac{1}{2} X^2 \quad x \quad = 0$$

$$\left| \begin{array}{c} X \\ \hline X_{IJ} \end{array} \right\} \left| \begin{array}{c} Z \\ \hline Z^J \end{array} \right\} = 0$$

AdS_5 from Projective space

X^{IJ} : skewsymmetric 4×4 matrix with projective invariance $X \rightarrow \lambda X$, parameterizing CP^5 .

Then the metric

$$ds^2 = \frac{dX^2}{X^2} + \frac{X \cdot dX}{X^2}{}^2$$

with

$$X^{IJ} = (X_b)^{IJ} + \frac{r^2}{2} I^{IJ}$$

becomes the AdS_5 metric (in Poincare co-ordinates),

$$ds^2 = \frac{dr^2 + dx \cdot dx}{r^2}$$

1.

" $\frac{X}{\frac{1}{2}X^2}$ parameterizes the boundary of AdS_5 . $[(X_b^{IJ})^2 = 0]$

2.

$$I^{IJ} = \begin{pmatrix} 0 & 0 \\ 0 & \dots \end{pmatrix}$$

Defining (Ambi-) Twistors

We now define the twistors (open subsets of \mathbb{CP}^3)

$$Z^I = (z^1; z^2) = (z^1; z^2; 1; z^2)$$

$$Y_J = (y^1; y^2) = (y^1; y^2; y^1; y^2)$$

These will play the role of ambitwistors for AdS_5 , but more generally twistor variables for the bulk AdS_5 .

Incidence relation on the boundary:

$$(X_b)_{IJ} Z^J = 0 \quad (x^1 = x^2)$$

$$X_b^{IJ} Y_J = 0 \quad (y^1 = x^1, y^2 = x^2)$$

These imply

$$C_b \quad Z^I Y_I^b = 0$$

Incidence relation in the bulk:

$$Z^I = X^{IJ} Y_J$$

$$\left(\begin{array}{c} x \\ \frac{1}{2}(x^2 + r^2) \end{array} \right) = \left(\begin{array}{c} x \\ y \end{array} \right)$$

This automatically satisfies the ambitwistor constraint:

$$C = Z^I Y_I = 0$$

which arises a fundamental gauge constraint [making quotient in $\mathfrak{psu}(2;2/4)_1$] from the worldsheet analysis.

[Adamo-Skinner-Williams'16]

As $r \neq 0$, the bulk AdS_5 incidence relations reduces to those on the boundary.

Twistor covering maps in AdS_5

To describe the string configurations which capture the dual $N = 4$ SYM, we promote the twistor variables and the AdS_5 spacetime as fields on the worldsheet:

$$X^{IJ}(z; \bar{z}); Z^I(z); Y_J(z); \mathbb{P}^I(\bar{z}); \mathbb{P}_J(\bar{z})$$

The AdS_5 incidence relations hold point-wise

$$Z^I(z) = X^{IJ}(z; \bar{z}) Y_J(z)$$

Twistor covering maps in AdS_5

For string configuration **near the boundary** of AdS_5

$$\begin{aligned} X^\mu(z) &= X^\mu(z; \bar{z}) \quad (z) \\ Y^\nu(z) &= X^\nu(z; \bar{z}) \quad Y^\nu(z) : \end{aligned}$$

Clearly then

$$\begin{aligned} \bar{\partial} X^\mu(z; \bar{z}) \quad (z) &= 0 ; \\ \bar{\partial} X^\nu(z; \bar{z}) \quad Y^\nu(z) &= 0 : \end{aligned}$$

so that

$$\bar{\partial} X^\mu(z; \bar{z}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \bar{\partial} \bar{V}(z) = \begin{pmatrix} 0 \\ \bar{V}(z) \end{pmatrix} \Rightarrow X^\mu(z; \bar{z}) = \begin{pmatrix} V(z) \\ 0 \end{pmatrix} \quad \bar{V}(z) :$$

Locally we can always view $X^\mu(z; \bar{z})$ as a holomorphic embedding into the boundary of an AdS_3 subspace of the bulk AdS_5 spacetime.

Maps to an AdS_3 subspace

We restrict to the kinematic set up where the boundary insertion points $f_{x_i} g$ lie within a two-dimensional plane. (Not a restriction for 2,3 and 4-point functions of $N = 4$ SYM), so that (globally) string lies on the boundary of the AdS_3 subspace.

$$X(z; \bar{z}) = \begin{pmatrix} V(z) & 0 \\ 0 & \bar{V}(\bar{z}) \end{pmatrix}; \quad (z) = \begin{pmatrix} 1(z) \\ 0 \end{pmatrix} \Rightarrow \quad \cdot(z) = \begin{pmatrix} V(z) & 1(z) \\ 0 & \end{pmatrix} :$$

$V(z) = 1(z) = 1(z)$ is a covering map from the genus zero worldsheet to the S^2 boundary of the AdS_3 . Generalizing the results for AdS_3 to this set up:

$$1(z) = \frac{R_{n-1}(z) Q_N^1(z)}{\prod_{i=1}^n (z - z_i)^{\frac{w_i}{2}}}; \quad 1(z) = \frac{R_{n-1}(z) P_N^1(z)}{\prod_{i=1}^n (z - z_i)^{\frac{w_i}{2}}} :$$

This ties up with $Z'(z) = \prod_{r=1}^p \frac{w_r - 1}{z - z_r} \frac{(z')_r}{(z - z_r)^{r+1=2}}$

Feynman covering

Correlator of n gauge-invariant scalar operators in the free $N = 4$ SYM:

$$\langle hO^{(w_1)}(x_1) \dots O^{(w_n)}(x_n) \rangle = \sum_{f_{n_{ij}g}} C_{f_{n_{ij}g}} \prod_{(i,j)} \frac{1}{x_{ij}^2} \quad ! \quad n_{ij}$$

For the two-point function joining $(x_i; x_j)$: $\frac{1}{x_{ij}^2}$, consider the following covering map

$$\Gamma(z) = \frac{V_j z^w + V_i}{z^w + 1} = \frac{P_w(z)}{Q_w(z)}$$

$$\text{with } V_k = x_k^{(1)} + ix_k^{(2)} .$$

Note that two points $(x_i; x_j)$ can always be taken to lie on a plane, corresponding states on the worldsheet are inserted at $z = 0$ and $z = 1$.

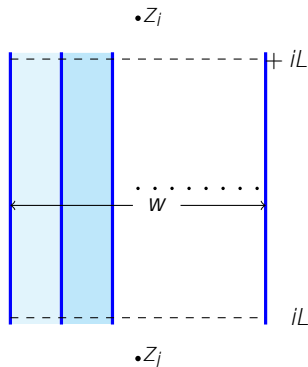
[Bhat-Gopakumar-PM-Radhakrishnan'21]

Feynman covering

It is convenient to view the covering map in u coordinate, where

$$z = \exp 2i \frac{u}{w}$$

mapping a vertical strip $(0 < \text{Re } u < w)$ onto the sphere such that $z = (0; 1)$ are images of $u = i1$, respectively, on the strip.



$$\Gamma(u) = V(z(u)) = \frac{V_i + V_j}{2} + \frac{V_i - V_j}{2i} \tan\left(\frac{u}{w}\right)$$

$$\Gamma(u) \quad V(z(u)) = \frac{V_i + V_j}{2} + \frac{V_i - V_j}{2i} \tan(\pi u)$$

This is essentially the unique map for which the Schwarzian is a constant:

$$S[\Gamma(u)] = \frac{\Gamma''''}{\Gamma'^3} = \frac{3}{2} \frac{\Gamma''^2}{\Gamma'^4} = 2^{-2} :$$

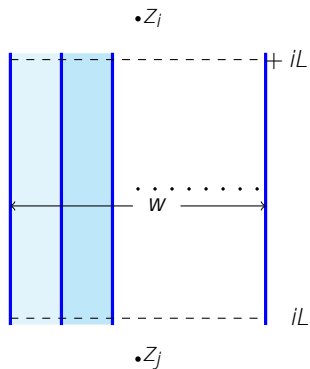
The unique Strebel quadratic differential on the strip with poles only at $u = i/2$ is also just du^2 . Thus

$$s(u) du^2 = \frac{1}{2} S[\Gamma(u)] du^2 :$$

This is a coordinate independent statement.

Feynman covering

The strip of width w is nothing other than the w double line edges glued together



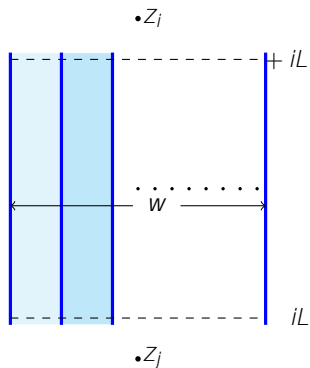
We can try to compute the Nambu-Goto area of the worldsheet in the "Strebel gauge":

$$ds^2 = j_s(z) dz d\bar{z}$$

Strebel area of the covering surface

In the u -coordinate, the "Strebel" area of the strip becomes $A_{ij} = 2LW$, where we've introduced cutoff in spacetime :

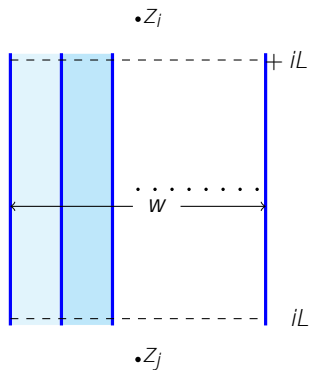
$$jV_i \quad \Gamma(u = iL)j = jV_j \quad \Gamma(u = -iL)j =$$



We find,

$$L = \frac{1}{4} \log \frac{x_{ij}^2}{2}$$

Covering surface area computes Feynman propagator

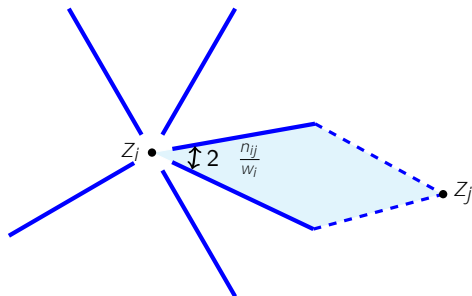


The Nambu-Goto weight with the "Strebel" area of the strip is then

$$\exp[2 A_{ij}] = \exp[2 \cdot 2LW] = 2^{2w} \frac{1}{x_{ij}^2} :$$

[Bhat-Gopakumar-PM-Radhakrishnan'21]

For multi-point correlator



Adding the areas of the strips near a vertex of the n-point correlator,

$$\langle h_{O_{w_1}}(x_1) \dots O_{w_n}(x_n) \rangle = \prod_{f_{n_{ij}g}} C_{f_{n_{ij}g}} \exp \left[2 \sum_{i < j}^{\#} A_{ij} \right]$$

[Bhat-Gopakumar-PM-Radhakrishnan'21]

Lessons (*take home message*)

1. Each Feynman graph can be associated to a point in the closed string moduli space, via the Strebel correspondence. Sum over all feynman diagrams defining the correlator goes over (in large twist limit) to an integral over this moduli space.
2. Worksheet fields can be seen as holomorphic covering map to the twistorial target space.

Outlook

1. To obtain the AdS_5 twistor incidence relation from worldsheet analysis.

[Gaberdiel-Gopakumar-PM-Knighton, in progress]

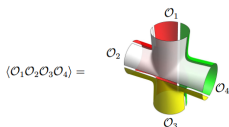
2. How does the Strebel differential deform away from the Schwarzian of covering map if we take $1=N$ corrections?

[Gopakumar-PM-Sarkar, in progress]

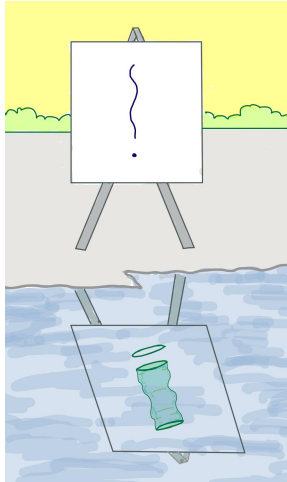
3. Towards the string dual of 2d Yang Mills ...

[Komatsu-PM, in progress]

4. Making connection with the Hexagonalization program of $N = 4$ SYM.



5. Making connection with the Mellin amplitudes of the perturbative $N = 4$ SYM.



Thanks for your attention