Correlation functions of scalar field theories from homotopy algebras

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1. Introduction

On-shell scattering amplitudes in quantum field theory

φ^3 theory

$$S = \int d^d x \left[-\frac{1}{2}\,\partial_\mu \varphi(x)\,\partial^\mu \varphi(x) - \frac{1}{2}\,m^2\,\varphi(x)^2 + \frac{1}{6}\,g\,\varphi(x)^3 \right]$$

$$\langle\, \varphi(x_1)\,\varphi(x_2)\,\varphi(x_1')\,\varphi(x_2')\,\rangle$$

$\downarrow$ LSZ reduction formula

$$\langle\, f \mid i \,\rangle = ig^2\,(2\pi)^d \delta^d(k_1 + k_2 - k_1' - k_2') \times \left[\frac{1}{(k_1+k_2)^2 + m^2} + \frac{1}{(k_1 - k_1')^2 + m^2} + \frac{1}{(k_1 - k_2')^2 + m^2} \right] + O(g^4)$$

On-shell scattering amplitudes in string theory

open bosonic string

$$\int_0^1 dt \, \langle \, cV_1(0) \, V_2(t) \, cV_3(1) \, cV_4(\infty) \, \rangle_{\text{disk}}$$

with

$$V_i(t) = \; : e^{ik_i X} : (t)$$

Open bosonic string field theory

$$S = \, - \, \frac{1}{2} \, \langle \, \Psi \, , Q\Psi \, \rangle - \frac{g}{3} \, \langle \, \Psi \, , \Psi * \Psi \, \rangle$$

$$\downarrow$$

$$\langle \, \Psi_1 * \Psi_2 \, , \frac{b_0}{L_0} \, (\, \Psi_3 * \Psi_4 \,) \, \rangle + \langle \, \Psi_4 * \Psi_1 \, , \frac{b_0}{L_0} \, (\, \Psi_2 * \Psi_3 \,) \, \rangle$$

When actions are written in terms of homotopy algebras such as A_∞ algebras and L_∞ algebras, expressions of on-shell scattering amplitudes in perturbation theory are universal for both string field theories and ordinary field theories.

$$\begin{array}{rl} \text{action} & \mathbf{Q} + \boldsymbol{m} \\ & \downarrow \\ \text{tree amplitude} & \mathbf{P}\,\boldsymbol{m}\,\dfrac{1}{\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m}}\,\mathbf{P} \\ \text{loop amplitude} & \mathbf{P}\,\boldsymbol{m}\,\dfrac{1}{\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m} + i\hbar\,\boldsymbol{h}\,\mathbf{U}}\,\mathbf{P} \end{array}$$

Explicit calculation of loop amplitudes are in general difficult in string theory.

We expect that homotopy algebras can be useful in gaining insights into quantum aspects of string field theories from ordinary field theories.

Digression

To provide a framework for proving the AdS/CFT correspondence, I want to construct a complete theory before taking the low-energy limit of D-branes.

We may think that such a theory would be open-closed superstring field theory, but my claim is that it can be described by open superstring field theory with the source term for gauge-invariant operators.
(See arXiv:2006.16449 for more details of this scenario.)

The long-standing problem of constructing an action involving the Ramond sector has been overcome in superstring field theory.

Kunitomo and Okawa, arXiv:1508.00366
Sen, arXiv:1508.05387

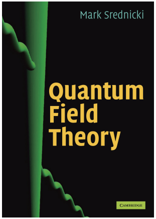
I am serious about quantizing open superstring field theory!

In addition to scattering amplitudes we find that correlation functions of scalar field theories can also be described in terms of homotopy algebras.

We explain explicit expressions for correlation functions of scalar field theories using quantum A_∞ algebras presented in arXiv:2203.05366.

Then we further discuss the application to the renormalization group.

Let us rewrite textbooks on quantum field theory in terms of homotopy algebras without using canonical quantization or the path integral.



The plan of the talk

1. Introduction
2. A_∞ algebra
3. Formula for correlation functions
4. Explicit calculations
5. Schwinger-Dyson equations
6. Renormalization group
7. Conclusions and discussion

2. A_∞ algebra

Open bosonic string field theory is described in terms of string field, which is a state of the boundary conformal field theory.

The Hilbert space $\mathcal{H}$ can be decomposed based on the ghost number as

$$\mathcal{H} = \ldots \oplus \mathcal{H}_{-1} \oplus \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \ldots ,$$

and the classical action is written in terms of Ψ in $\mathcal{H}_1$.

Consider an action of the form:

$$S = -\frac{1}{2} \langle \Psi, V_1(\Psi) \rangle - \frac{g}{3} \langle \Psi, V_2(\Psi, \Psi) \rangle - \frac{g^2}{4} \langle \Psi, V_3(\Psi, \Psi, \Psi) \rangle + O(g^3) ,$$

where $\langle A_1, A_2 \rangle$ is the BPZ inner product of A_1 and A_2, V_n is an n-string product, and g is the string coupling constant.

This action is invariant up to $O(g^3)$ under the gauge transformation with the gauge parameter Λ in $\mathcal{H}_0$ given by

$$\delta_\Lambda \Psi = V_1(\Lambda) + g\,(\,V_2(\Psi,\Lambda) - V_2(\Lambda,\Psi)\,)$$
$$+\,g^2\,(\,V_3(\Psi,\Psi,\Lambda) - V_3(\Psi,\Lambda,\Psi) + V_3(\Lambda,\Psi,\Psi)\,) + O(g^3)$$

if the multi-string products satisfy the following relations:

$$V_1(V_1(A_1)) = 0\,,$$
$$V_1(V_2(A_1,A_2)) - V_2(V_1(A_1),A_2) - (-1)^{A_1} V_2(A_1,V_1(A_2)) = 0\,,$$
$$V_1(V_3(A_1,A_2,A_3)) + V_3(V_1(A_1),A_2,A_3)$$
$$+\,(-1)^{A_1} V_3(A_1,V_1(A_2),A_3) + (-1)^{A_1+A_2} V_3(A_1,A_2,V_1(A_3))$$
$$-\,V_2(V_2(A_1,A_2),A_3) + V_2(A_1,V_2(A_2,A_3)) = 0\,.$$

These relations can be extended to higher orders and called A_∞ relations. (In this talk all the discussions on cyclic properties are omitted.)

In open bosonic string field theory constructed by Witten, V_1 and V_2 are given by the BRST operator Q and the star product as

$$V_1(A_1) = QA_1\,, \qquad V_1(A_1, A_2) = A_1 * A_2\,,$$

and the A_∞ relations are satisfied without higher products because of the following properties

$$\begin{aligned} Q^2 &= 0\,, \\ Q\,(A_1 * A_2) &= QA_1 * A_2 + (-1)^{A_1} A_1 * QA_2\,, \\ A_1 * (A_2 * A_3) &= (A_1 * A_2) * A_3\,. \end{aligned}$$

A set of multi-string products satisfying A_∞ relations for the open string and L_∞ relations for the closed string can be constructed from an appropriate decomposition of the moduli space of Riemann surfaces.

hep-th/9206084, Zwiebach

Let us simplify the description of A_∞ relations in three steps.

Step 1: Degree

We introduce *degree* defined by

$$\deg(A) = \epsilon(A) + 1 \mod 2\,,$$

where $\epsilon(A)$ is the Grassmann parity of A, and we define

$$\begin{aligned}
\omega(A_1, A_2) &= (-1)^{\deg(A_1)} \langle\, A_1, A_2 \,\rangle\,, \\
M_1(A_1) &= V_1(A_1)\,, \\
M_2(A_1, A_2) &= (-1)^{\deg(A_1)}\, V_2(A_1, A_2)\,, \\
M_3(A_1, A_2, A_3) &= (-1)^{\deg(A_2)}\, V_3(A_1, A_2, A_3)\,, \\
&\ \ \vdots
\end{aligned}$$

The A_∞ relations are written as

$$
\begin{aligned}
& M_1(M_1(A_1)) = 0\,, \\
& M_1(M_2(A_1,A_2)) + M_2(M_1(A_1),A_2) + (-1)^{\deg(A_1)} M_2(A_1, M_1(A_2)) = 0\,, \\
& M_1(M_3(A_1,A_2,A_3)) + M_3(M_1(A_1),A_2,A_3) \\
& \; + (-1)^{\deg(A_1)} M_3(A_1, M_1(A_2), A_3) + (-1)^{\deg(A_1)+\deg(A_2)} M_3(A_1, A_2, M_1(A_3)) \\
& \; + M_2(M_2(A_1,A_2),A_3) + (-1)^{\deg(A_1)} M_2(A_1, M_2(A_2,A_3)) = 0\,, \\
& \qquad\qquad\qquad\qquad \vdots
\end{aligned}
$$

The Grassmann parities of V_n are not uniform, but the M_n are all degree-odd.

Step 2: Tensor products of $\mathcal{H}$

We denote the tensor product of n copies of $\mathcal{H}$ by $\mathcal{H}^{\otimes n}$. For an n-string product $c_n(A_1, A_2, \ldots, A_n)$ we define a corresponding operator c_n which maps $\mathcal{H}^{\otimes n}$ into $\mathcal{H}$ by

$$c_n\,(A_1 \otimes A_2 \otimes \ldots \otimes A_n) \equiv c_n(A_1, A_2, \ldots, A_n)\,.$$

We also introduce the vector space for the zero-string space denoted by $\mathcal{H}^{\otimes 0}$. It is a one-dimensional vector space given by multiplying a single basis vector $\mathbf{1}$ by complex numbers. The vector $\mathbf{1}$ satisfies

$$\mathbf{1} \otimes A = A\,, \qquad A \otimes \mathbf{1} = A$$

for any string field A.

Example

$$M_1(M_2(A_1, A_2)) + M_2(M_1(A_1), A_2) + (-1)^{\deg(A_1)} M_2(A_1, M_1(A_2)) = 0$$

$$\downarrow$$

$$M_1\, M_2\, (A_1 \otimes A_2) + M_2\, (M_1 A_1 \otimes A_2) + (-1)^{\deg(A_1)}\, M_2\, (A_1 \otimes M_1 A_2) = 0$$

We denote the identity map from $\mathcal{H}$ to $\mathcal{H}$ by $\mathbb{I}$, and we write

$$M_1 A_1 \otimes A_2 = (\, M_1 \otimes \mathbb{I}\,)\, (A_1 \otimes A_2)\,,$$

$$(-1)^{\deg(A_1)} A_1 \otimes M_1 A_2 = (\, \mathbb{I} \otimes M_1\,)\, (A_1 \otimes A_2)\,.$$

We then have

$$M_1\, M_2\, (A_1 \otimes A_2) + M_2\, (M_1 A_1 \otimes A_2) + (-1)^{\deg(A_1)}\, M_2\, (A_1 \otimes M_1 A_2) = 0$$

$$\downarrow$$

$$M_1\, M_2\, (A_1 \otimes A_2) + M_2\, (\, M_1 \otimes \mathbb{I}\,)\, (A_1 \otimes A_2) + M_2\, (\, \mathbb{I} \otimes M_1\,)\, (A_1 \otimes A_2) = 0$$

$$\downarrow$$

$$M_1\, M_2 + M_2\, (\, M_1 \otimes \mathbb{I}\,) + M_2\, (\, \mathbb{I} \otimes M_1\,) = 0$$

A_∞ relations

$$
\begin{aligned}
&M_1\, M_1 = 0\,,\\
&M_1\, M_2 + M_2\,(\,M_1 \otimes \mathbb{I} + \mathbb{I} \otimes M_1\,) = 0\,,\\
&M_1\, M_3 + M_3\,(\,M_1 \otimes \mathbb{I} \otimes \mathbb{I} + \mathbb{I} \otimes M_1 \otimes \mathbb{I} + \mathbb{I} \otimes \mathbb{I} \otimes M_1\,)\\
&+ M_2\,(\,M_2 \otimes \mathbb{I} + \mathbb{I} \otimes M_2\,) = 0\,,\\
&\qquad\qquad\vdots
\end{aligned}
$$

Step 3: Coderivations

It is convenient to consider linear operators acting on the vector space $T\mathcal{H}$ defined by

$$T\mathcal{H} = \mathcal{H}^{\otimes 0} \oplus \mathcal{H} \oplus \mathcal{H}^{\otimes 2} \oplus \mathcal{H}^{\otimes 3} \oplus \ldots .$$

We denote the projection operator onto $\mathcal{H}^{\otimes n}$ by π_n.

For a map c_n from $\mathcal{H}^{\otimes n}$ to $\mathcal{H}$, we define an associated operator $\boldsymbol{c}_n$ acting on $T\mathcal{H}$ as follows.

$$\begin{aligned}
\boldsymbol{c}_n \, \pi_m &= 0 \quad \text{for} \quad m < n \,, \\
\boldsymbol{c}_n \, \pi_n &= c_n \, \pi_n \,, \\
\boldsymbol{c}_n \, \pi_{n+1} &= (\, c_n \otimes \mathbb{I} + \mathbb{I} \otimes c_n \,) \, \pi_{n+1} \,, \\
\boldsymbol{c}_n \, \pi_{n+2} &= (\, c_n \otimes \mathbb{I} \otimes \mathbb{I} + \mathbb{I} \otimes c_n \otimes \mathbb{I} + \mathbb{I} \otimes \mathbb{I} \otimes c_n \,) \, \pi_{n+2} \,, \\
&\;\;\vdots
\end{aligned}$$

An operator acting on $T\mathcal{H}$ of this form is called a *coderivation*.

We define $\mathbf{M}$ by

$$\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3 + \ldots$$

for coderivations $\mathbf{M}_n$ associated with M_n. Then the A_∞ relations can be compactly expressed as

$$\mathbf{M}^2 = 0\,.$$

The information on the action can be encoded in a degree-odd coderivation which squares to zero. Conversely, we can construct a gauge-invariant action from a degree-odd coderivation which squares to zero.

When we consider projections onto subspaces of $\mathcal{H}$, homotopy algebras have turned out to provide useful tools.

- Projection onto on-shell states $\to$ on-shell scattering amplitudes
  Kajiura, math/0306332

- Projection onto the physical sector
  $\to$ mapping between covariant and light-cone string field theories
  Erler and Matsunaga, arXiv:2012.09521

- Projection onto the massless sector $\to$ the low-energy effective action
  Sen, arXiv:1609.00459
  Erbin, Maccaferri, Schnabl and Vošmera, arXiv:2006.16270
  Koyama, Okawa and Suzuki, arXiv:2006.16710

Let us decompose $\mathbf{M}$ as

$$\mathbf{M} = \mathbf{Q} + \boldsymbol{m}\,,$$

where $\mathbf{Q}$ describes the free theory and $\boldsymbol{m}$ is for interactions. We consider projections which commute with Q.

We denote the projection operator by P:

$$P^2 = P \,, \qquad P\,Q = Q\,P \,.$$

We then promote P on $\mathcal{H}$ to $\mathbf{P}$ on $T\mathcal{H}$ as follows:

$$\begin{aligned}
\mathbf{P}\,\pi_0 &= \pi_0 \,, \\
\mathbf{P}\,\pi_1 &= P\,\pi_1 \,, \\
\mathbf{P}\,\pi_2 &= (\,P \otimes P\,)\,\pi_2 \,, \\
\mathbf{P}\,\pi_3 &= (\,P \otimes P \otimes P)\,\pi_3 \,, \\
&\vdots
\end{aligned}$$

The operators $\mathbf{Q}$ and $\mathbf{P}$ satisfy

$$\mathbf{P}^2 = \mathbf{P} \,, \qquad \mathbf{Q}\,\mathbf{P} = \mathbf{P}\,\mathbf{Q} \,.$$

In the context of the projection onto the massless sector, the propagator h for massive fields is given by

$$h = \frac{b_0}{L_0}\,(\,\mathbb{I} - P\,)\,.$$

In general we consider h satisfying the following relations:

$$Q\,h + h\,Q = \mathbb{I} - P\,, \qquad h\,P = 0\,, \qquad P\,h = 0\,, \qquad h^2 = 0\,.$$

We then promote h on $\mathcal{H}$ to $\boldsymbol{h}$ on $T\mathcal{H}$ as follows:

$$\begin{aligned}
\boldsymbol{h}\,\pi_0 &= 0\,,\\
\boldsymbol{h}\,\pi_1 &= h\,\pi_1\,,\\
\boldsymbol{h}\,\pi_2 &= (\,h\otimes P + \mathbb{I}\otimes h\,)\,\pi_2\,,\\
\boldsymbol{h}\,\pi_3 &= (\,h\otimes P\otimes P + \mathbb{I}\otimes h\otimes P + \mathbb{I}\otimes\mathbb{I}\otimes h\,)\,\pi_3\,,\\
&\ \ \vdots
\end{aligned}$$

The relations involving Q, P, and h are promoted to the following relations

$$\mathbf{Q}\,\boldsymbol{h} + \boldsymbol{h}\,\mathbf{Q} = \mathbf{I} - \mathbf{P}\,, \qquad \boldsymbol{h}\,\mathbf{P} = 0\,, \qquad \mathbf{P}\,\boldsymbol{h} = 0\,, \qquad \boldsymbol{h}^2 = 0\,,$$

where $\mathbf{I}$ is the identity operator on $T\mathcal{H}$.

The important point is that the theory after the projection inherits the A_∞ structure from the theory before the projection as follows:

$$\mathbf{Q} + \boldsymbol{m} \quad \to \quad \mathbf{P}\,\mathbf{Q}\,\mathbf{P} + \mathbf{P}\,\boldsymbol{m}\,\frac{1}{\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m}}\,\mathbf{P}\,,$$

which is known as *homological perturbation lemma*.

3. Formula for correlation functions

Let us consider φ^3 theory in d dimensions:

$$S = \int d^d x \left[-\frac{1}{2} \partial_\mu \varphi(x) \, \partial^\mu \varphi(x) - \frac{1}{2} \, m^2 \, \varphi(x)^2 + \frac{1}{6} \, g \, \varphi(x)^3 \right].$$

To describe this action in terms of an A_∞ algebra, we introduce two copies of the vector space of functions of x. We denote them by $\mathcal{H}_1$ and $\mathcal{H}_2$, and $\mathcal{H}$ is given by

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \, .$$

We define ω, Q, and b_2 by

$$\begin{aligned}
\omega \, (\, \varphi_1(x), \varphi_2(x) \,) &= \int d^d x \, \varphi_1(x) \, \varphi_2(x) \quad \text{for} \quad \varphi_1(x) \in \mathcal{H}_1 \, , \, \varphi_2(x) \in \mathcal{H}_2 \, , \\
Q \, \varphi(x) &= (\, -\partial^2 + m^2 \,) \, \varphi(x) \in \mathcal{H}_2 \quad \text{for} \quad \varphi(x) \in \mathcal{H}_1 \, , \\
b_2 \, (\, \varphi_1(x) \otimes \varphi_2(x) \,) &= -\frac{g}{2} \, \varphi_1(x) \, \varphi_2(x) \in \mathcal{H}_2 \quad \text{for} \quad \varphi_1(x) \, , \varphi_2(x) \in \mathcal{H}_1 \, .
\end{aligned}$$

The A_∞ structure of the classical action is described by $\mathbf{Q} + \boldsymbol{b}_2$. The A_∞ relations are trivially satisfied for this theory without gauge symmetries.

When we consider on-shell scattering amplitudes, we use the projection onto on-shell states. The action of h on $\varphi(x)$ in $\mathcal{H}_2$ is given by

$$h\,\varphi(x) = \int d^d y \int \frac{d^d p}{(2\pi)^d}\,\frac{e^{-ip\,(x-y)}}{p^2 + m^2 - i\epsilon}\,\varphi(y)\,.$$

In the case of the projection onto on-shell states, $\mathbf{P}\,\mathbf{Q}\,\mathbf{P}$ vanishes and on-shell scattering amplitudes at the tree level can be calculated from

$$\mathbf{P}\,\boldsymbol{b}_2\,\frac{1}{\mathbf{I} + \boldsymbol{h}\,\boldsymbol{b}_2}\,\mathbf{P}\,.$$

When we discuss the quantum theory, we need to include conterterms. We denote the coderivation after including counterterms by $\mathbf{Q} + \boldsymbol{m}$. On-shell scattering amplitudes including loop diagrams can be calculated from

$$\mathbf{P}\, \boldsymbol{m}\, \frac{1}{\mathbf{I} + \boldsymbol{h}\, \boldsymbol{m} + i\hbar\, \boldsymbol{h}\, \mathbf{U}}\, \mathbf{P}\,.$$

The operator $\mathbf{U}$ consists of maps from $\mathcal{H}^{\otimes n}$ to $\mathcal{H}^{\otimes (n+2)}$. When the vector space $\mathcal{H}$ is given by $\mathcal{H}_1 \oplus \mathcal{H}_2$, the operator $\mathbf{U}$ incorporates a pair of basis vectors of $\mathcal{H}_1$ and $\mathcal{H}_2$. We denote the basis vector of $\mathcal{H}_1$ by e^α, where α is the label of the basis vectors. For $\mathcal{H}_2$ we denote the basis vector by e_α, and repeated indices are implicitly summed over. We use the following choice for e^α and e_α:

$$\ldots \otimes e^\alpha \otimes \ldots \otimes e_\alpha \otimes \ldots = \int \frac{d^d p}{(2\pi)^d}\, \ldots \otimes e^{-ipx} \otimes \ldots \otimes e^{ipx} \otimes \ldots\,.$$

The action of $\mathbf{U}$ on $\mathcal{H}^{\otimes 0}$ is given by

$$\mathbf{U}\,\mathbf{1} = e^\alpha \otimes e_\alpha + e_\alpha \otimes e^\alpha\,,$$

and the action of $\mathbf{U}$ on $\mathcal{H}$ is given by

$$\begin{aligned}\mathbf{U}\,\varphi(x) = {} & e^\alpha \otimes e_\alpha \otimes \varphi(x) + (-1)^{\deg(\varphi)} e^\alpha \otimes \varphi(x) \otimes e_\alpha \\ & + (-1)^{\deg(\varphi)} \varphi(x) \otimes e^\alpha \otimes e_\alpha + e_\alpha \otimes e^\alpha \otimes \varphi(x) \\ & + e_\alpha \otimes \varphi(x) \otimes e^\alpha + (-1)^{\deg(\varphi)} \varphi(x) \otimes e_\alpha \otimes e^\alpha\,.\end{aligned}$$

A_∞ algebras are extended to quantum A_∞ algebras in the quantum theory. The quantum A_∞ relations are again trivially satisfied for this theory without gauge symmetries.

If we recall that the projection onto the massless sector corresponds to integrating out massive fields, carrying out the path integral *completely* should correspond to the projection with

$$P = 0\,.$$

The associated operator $\mathbf{P}$ corresponds to the projection onto $\mathcal{H}^{\otimes 0}$:

$$\mathbf{P} = \pi_0\,.$$

This may result in a trivial theory at the classical case, but it can be nontrivial for the quantum case and in fact it is exactly what we do when we calculate correlation functions.

Let us consider scalar field theories in Euclidean space. We define $\boldsymbol{f}$ by

$$\boldsymbol{f} = \frac{1}{\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m} - \boldsymbol{h}\,\mathbf{U}}\,,$$

which corresponds to

$$\frac{1}{\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m} + i\hbar\,\boldsymbol{h}\,\mathbf{U}}$$

in Minkowski space.

While $\mathbf{P}\,\boldsymbol{m}\,\boldsymbol{f}\,\mathbf{P}$ vanishes, $\boldsymbol{f}$ is nonvanishing and this operator plays a central role in generating Feynman diagrams.

We claim that information on correlation functions is encoded in $\boldsymbol{f}\,\mathbf{1}$ associated with the case where $P = 0$.

More explicitly, correlation functions are given by

$$\langle\, \varphi(x_1)\, \varphi(x_2)\ \ldots\ \varphi(x_n)\,\rangle$$
$$= \omega_n\,(\,\pi_n\,\boldsymbol{f}\,\mathbf{1}\,,\delta^d(x-x_1)\otimes\delta^d(x-x_2)\otimes\ldots\otimes\delta^d(x-x_n)\,)\,,$$

where

$$\omega_n\,(\,\varphi_1(x)\otimes\varphi_2(x)\otimes\ldots\otimes\varphi_n(x)\,,\varphi_1'(x)\otimes\varphi_2'(x)\otimes\ldots\otimes\varphi_n'(x)\,)$$
$$= \prod_{i=1}^{n}\omega\,(\,\varphi_i(x)\,,\varphi_i'(x)\,)\,.$$

The formula may look complicated, but it states that $\pi_n\,\boldsymbol{f}\,\mathbf{1}$ gives the n-point function by simply replacing x with x_i in the i-th sector in $\mathcal{H}^{\otimes n}$.

For example, when $\pi_3 \, \boldsymbol{f} \, \mathbf{1}$ takes the form

$$\pi_3 \, \boldsymbol{f} \, \mathbf{1} = \sum_a f_a(x) \otimes g_a(x) \otimes h_a(x) \, ,$$

the three-point function is given by

$$\begin{aligned} & \langle \, \varphi(x_1) \, \varphi(x_2) \, \varphi(x_3) \, \rangle \\ & = \omega_3 \, (\, \pi_3 \, \boldsymbol{f} \, \mathbf{1} \, , \delta^d(x - x_1) \otimes \delta^d(x - x_2) \otimes \delta^d(x - x_3) \,) \\ & = \sum_a f_a(x_1) \, g_a(x_2) \, h_a(x_3) \, . \end{aligned}$$

This can be summarized as the following replacement rule:

$$\pi_3 \, \boldsymbol{f} \, \mathbf{1} = \sum_a f_a(x) \otimes g_a(x) \otimes h_a(x)$$

$$\downarrow$$

$$\langle \, \varphi(x_1) \, \varphi(x_2) \, \varphi(x_3) \, \rangle = \sum_a f_a(x_1) \, g_a(x_2) \, h_a(x_3) \, .$$

4. Explicit calculations

Let us first demonstrate that correlation functions of the free theory are correctly reproduced. We denote correlation functions of the free theory by $\langle\, \varphi(x_1)\, \varphi(x_2)\, \ldots\, \varphi(x_n)\, \rangle^{(0)}$. In this case $\boldsymbol{f}\,\mathbf{1}$ is given by

$$\boldsymbol{f}\,\mathbf{1} = \frac{1}{\mathbf{I} - \boldsymbol{h}\,\mathbf{U}}\,\mathbf{1}\,.$$

The action of h on $\varphi(x)$ in $\mathcal{H}_2$ is given by

$$h\,\varphi(x) = \int d^d y \int \frac{d^d p}{(2\pi)^d}\, \frac{e^{-ip\,(x-y)}}{p^2+m^2}\, \varphi(y)\,.$$

The two-point function can be calculated from $\pi_2\,\boldsymbol{f}\,\mathbf{1}$. We find

$$\pi_2\,\boldsymbol{f}\,\mathbf{1} = \pi_2\,\boldsymbol{h}\,\mathbf{U}\,\mathbf{1} = e^\alpha \otimes h\, e_\alpha = \int \frac{d^d p}{(2\pi)^d}\, e^{-ipx} \otimes \frac{1}{p^2+m^2}\, e^{ipx}\,.$$

Following the replacement rule, the two-point function is given by

$$\begin{aligned}\langle\, \varphi(x_1)\, \varphi(x_2)\, \rangle^{(0)} &= \omega_2\,(\,\pi_2\,\boldsymbol{f}\,\mathbf{1}\,, \delta^d(x-x_1) \otimes \delta^d(x-x_2)\,)\\ &= \int \frac{d^d p}{(2\pi)^d}\, \frac{e^{-ip\,(x_1-x_2)}}{p^2+m^2}\,.\end{aligned}$$

The four-point function can be calculated from $\pi_4 \, \boldsymbol{f} \, \mathbf{1}$. We fikd

$$\begin{aligned}\pi_4 \, \boldsymbol{f} \, \mathbf{1} &= \pi_4 \, \boldsymbol{h} \, \mathbf{U} \, \boldsymbol{h} \, \mathbf{U} \, \mathbf{1} \\ &= e^\beta \otimes e^\alpha \otimes h \, e_\alpha \otimes h \, e_\beta + e^\alpha \otimes e^\beta \otimes h \, e_\alpha \otimes h \, e_\beta \\ &\quad + e^\alpha \otimes h \, e_\alpha \otimes e^\beta \otimes h \, e_\beta \, .\end{aligned}$$

The first term on the right-hand side is given by

$$\begin{aligned}& e^\beta \otimes e^\alpha \otimes h \, e_\alpha \otimes h \, e_\beta \\ & = \int \frac{d^d p_1}{(2\pi)^d} \int \frac{d^d p_2}{(2\pi)^d} \, e^{-ip_2 x} \otimes e^{-ip_1 x} \otimes \frac{1}{p_1^2 + m^2} \, e^{ip_1 x} \otimes \frac{1}{p_2^2 + m^2} \, e^{ip_2 x} \, ,\end{aligned}$$

and the contribution to the four-point function is

$$\begin{aligned}& \int \frac{d^d p_1}{(2\pi)^d} \int \frac{d^d p_2}{(2\pi)^d} \, e^{-ip_2 x_1} \, e^{-ip_1 x_2} \, \frac{1}{p_1^2 + m^2} \, e^{ip_1 x_3} \, \frac{1}{p_2^2 + m^2} \, e^{ip_2 x_4} \\ & = \int \frac{d^d p_1}{(2\pi)^d} \, \frac{e^{-ip_1 \, (x_2 - x_3)}}{p_1^2 + m^2} \int \frac{d^d p_2}{(2\pi)^d} \, \frac{e^{-ip_2 \, (x_1 - x_4)}}{p_2^2 + m^2} \\ & = \langle \, \varphi(x_2) \, \varphi(x_3) \, \rangle^{(0)} \, \langle \, \varphi(x_1) \, \varphi(x_4) \, \rangle^{(0)} \, .\end{aligned}$$

The second and third terms can be calculated similarly, and the four-point function is given by

$$
\begin{aligned}
&\langle\, \varphi(x_1)\, \varphi(x_2)\, \varphi(x_3)\, \varphi(x_4)\, \rangle^{(0)} \\
&= \omega_4\, (\, \pi_4\, \boldsymbol{f}\, \mathbf{1}\, , \delta^d(x - x_1) \otimes \delta^d(x - x_2) \otimes \delta^d(x - x_3) \otimes \delta^d(x - x_4)\,) \\
&= \langle\, \varphi(x_2)\, \varphi(x_3)\, \rangle^{(0)} \langle\, \varphi(x_1)\, \varphi(x_4)\, \rangle^{(0)} + \langle\, \varphi(x_1)\, \varphi(x_3)\, \rangle^{(0)} \langle\, \varphi(x_2)\, \varphi(x_4)\, \rangle^{(0)} \\
&\qquad + \langle\, \varphi(x_1)\, \varphi(x_2)\, \rangle^{(0)} \langle\, \varphi(x_3)\, \varphi(x_4)\, \rangle^{(0)}\, .
\end{aligned}
$$

We have thus reproduced Wick's theorem for four-point functions, and it is not difficult to extend the analysis to six-point functions and further.

Let us next consider φ^3 theory. The action including counterterms is given by

$$S = \int d^d x \left[\frac{1}{2} Z_\varphi \, \partial_\mu \varphi(x) \, \partial_\mu \varphi(x) + \frac{1}{2} Z_m \, m^2 \, \varphi(x)^2 - \frac{1}{6} Z_g \, g \, \varphi(x)^3 - Y \varphi(x) \right],$$

where Y, Z_φ, Z_m, and Z_g are constants.

The operators m_0, m_1, and m_2 for this action are defined by

$$\begin{aligned} m_0 \, \mathbf{1} &= -Y \,, \\ m_1 \, \varphi(x) &= -(Z_\varphi - 1) \, \partial^2 \varphi(x) + (Z_m - 1) \, m^2 \, \varphi(x) \,, \\ m_2 \, (\varphi_1(x) \otimes \varphi_2(x)) &= -\frac{g}{2} \, Z_g \, \varphi_1(x) \, \varphi_2(x) \,. \end{aligned}$$

The coderivations corresponding to m_0, m_1, and m_2 are denoted by $\boldsymbol{m}_0$, $\boldsymbol{m}_1$, and $\boldsymbol{m}_2$, and we define $\boldsymbol{m}$ by

$$\boldsymbol{m} = \boldsymbol{m}_0 + \boldsymbol{m}_1 + \boldsymbol{m}_2 \,.$$

Let us calculate correlation functions in perturbation theory with respect to g. We expand Y, Z_φ, Z_m, and Z_g in g as follows:

$$
\begin{aligned}
Y &= g\, Y^{(1)} + O(g^3)\,, \\
Z_\varphi &= 1 + g^2 Z_\varphi^{(1)} + O(g^4)\,, \\
Z_m &= 1 + g^2 Z_m^{(1)} + O(g^4)\,, \\
Z_g &= 1 + g^2 Z_g^{(1)} + O(g^4)\,.
\end{aligned}
$$

Correspondingly, we expand $\boldsymbol{m}_0$, $\boldsymbol{m}_1$, and $\boldsymbol{m}_2$ in g as

$$
\boldsymbol{m}_0 = \sum_{\ell=0}^{\infty} \boldsymbol{m}_0^{(\ell)}\,, \qquad \boldsymbol{m}_1 = \sum_{\ell=0}^{\infty} \boldsymbol{m}_1^{(\ell)}\,, \qquad \boldsymbol{m}_2 = \sum_{\ell=0}^{\infty} \boldsymbol{m}_2^{(\ell)}\,,
$$

where $\boldsymbol{m}_n^{(\ell)}$ is of $O(g^{n-1+2\ell})$.

We also expand $\boldsymbol{m}$ in g as

$$\boldsymbol{m} = \sum_{\ell=0}^{\infty} \boldsymbol{m}^{(\ell)}\,,$$

where

$$\boldsymbol{m}^{(\ell)} = \boldsymbol{m}_0^{(\ell)} + \boldsymbol{m}_1^{(\ell)} + \boldsymbol{m}_2^{(\ell)}\,.$$

The coderivation $\boldsymbol{m}^{(0)}$ describes the interaction of the classical action and is given by

$$\boldsymbol{m}^{(0)} = \boldsymbol{b}_2\,.$$

The coderivation $\boldsymbol{m}^{(1)}$ describes counterterms at one loop, and $m_0^{(1)}$, $m_1^{(1)}$, and $m_2^{(1)}$ are given by

$$\begin{aligned} m_0^{(1)}\,\mathbf{1} &= \,-\,g\,Y^{(1)}\,, \\ m_1^{(1)}\,\varphi(x) &= \,-\,g^2 Z_\varphi^{(1)}\,\partial^2\varphi(x) + g^2 Z_m^{(1)}\,m^2\,\varphi(x)\,, \\ m_2^{(1)}\,(\,\varphi_1(x)\otimes\varphi_2(x)\,) &= \,-\,\frac{g^3}{2}\,Z_g^{(1)}\,\varphi_1(x)\,\varphi_2(x)\,. \end{aligned}$$

The one-point function can be calculated from $\pi_1 \, \boldsymbol{f} \, \mathbf{1}$.

$$\begin{aligned}
\pi_1 \, \boldsymbol{f} \, \mathbf{1} = & \; - \pi_1 \, \boldsymbol{h} \, \boldsymbol{b}_2 \, \boldsymbol{h} \, \mathbf{U} \, \mathbf{1} - \pi_1 \, \boldsymbol{h} \, \boldsymbol{m}_0^{(1)} \, \mathbf{1} + O(g^2) \\
= & \; - h \, b_2 \, (\, e^\alpha \otimes h \, e_\alpha \,) - h \, m_0^{(1)} \, \mathbf{1} + O(g^2) \, .
\end{aligned}$$

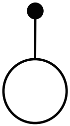
The explicit form of the terms of $O(g)$ is given by

$$- h \, b_2 \, (\, e^\alpha \otimes h \, e_\alpha \,) - h \, m_0^{(1)} \, \mathbf{1} = \frac{g}{m^2} \left[\, \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \, \frac{1}{p^2 + m^2} + Y^{(1)} \, \right] ,$$

and the one-point function is given by

$$\begin{aligned}
\langle \, \varphi(x_1) \, \rangle &= \omega_1 \, (\, \pi_1 \, \boldsymbol{f} \, \mathbf{1} \, , \delta^d(x - x_1) \,) \\
&= \frac{g}{m^2} \left[\, \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \, \frac{1}{p^2 + m^2} + Y^{(1)} \, \right] + O(g^2) \, .
\end{aligned}$$

We have reproduced the contribution from the one-loop tadpole diagram:



Note that the correct symmetry factor appeared. While we can make the one-point function vanish at $O(g)$ by choosing $Y^{(1)}$ to cancel the contribution from the one-loop tadpole diagram, we leave it finite and keep track of the appearance of one-loop tadpoles.

We define $\Gamma_0^{(1)}$ by

$$\Gamma_0^{(1)}\,\mathbf{1} = b_2\,(\,e^\alpha \otimes h\,e_\alpha\,) + m_0^{(1)}\,\mathbf{1} = \,-\,\frac{g}{2}\int \frac{d^dp}{(2\pi)^d}\,\frac{1}{p^2+m^2} - g\,Y^{(1)}\,.$$

The operator $\Gamma_0^{(1)}$ describes the linear term at one loop in the 1PI effective action. We write the one-point function as

$$\langle\,\varphi(x_1)\,\rangle = \langle\,\varphi(x_1)\,\rangle^{(1)} + O(g^2)\,,$$

where

$$\begin{aligned}\langle\,\varphi(x_1)\,\rangle^{(1)} &= \,-\,\omega_1\,(\,h\,\Gamma_0^{(1)}\,\mathbf{1}\,,\delta^d(x-x_1)\,)\\ &= \frac{g}{m^2}\left[\,\frac{1}{2}\int \frac{d^dp}{(2\pi)^d}\,\frac{1}{p^2+m^2} + Y^{(1)}\right].\end{aligned}$$

The two-point function can be calculated from $\pi_2 \, \boldsymbol{f} \, \mathbf{1}$. We find

$$\begin{aligned}
\pi_2 \, \boldsymbol{f} \, \mathbf{1} = e^\alpha \otimes h \, e_\alpha - e^\alpha \otimes h \, \Gamma_1^{(1)} \, h \, e_\alpha \\
+ e^\alpha \otimes h \, b_2 \, (\, h \, e_\alpha \otimes h \, \Gamma_0^{(1)} \, \mathbf{1} \,) + e^\alpha \otimes h \, b_2 \, (\, h \, \Gamma_0^{(1)} \, \mathbf{1} \otimes h \, e_\alpha \,) \\
+ h \, \Gamma_0^{(1)} \, \mathbf{1} \otimes h \, \Gamma_0^{(1)} \, \mathbf{1} + O(g^3) \, ,
\end{aligned}$$

where the action of $\Gamma_1^{(1)}$ on e^{ikx} in $\mathcal{H}_1$ is given by

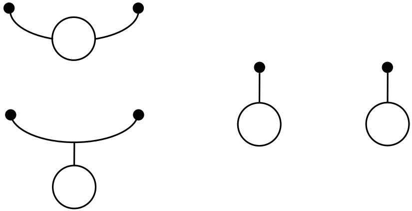
$$\begin{aligned}
&\Gamma_1^{(1)} \, e^{ikx} \\
&= \Big[- \frac{g^2}{2} \int \frac{d^d p}{(2\pi)^d} \, \frac{1}{(p+k)^2 + m^2} \, \frac{1}{p^2 + m^2} + g^2 \, Z_\varphi^{(1)} \, k^2 + g^2 \, Z_m^{(1)} \, m^2 \Big] \, e^{ikx} \, .
\end{aligned}$$

The two-point function is given by

$$\begin{aligned}
\langle \, \varphi(x_1) \, \varphi(x_2) \, \rangle &= \omega_2 \, (\, \pi_2 \, \boldsymbol{f} \, \mathbf{1} \, , \delta^d(x - x_1) \otimes \delta^d(x - x_2) \,) \\
&= \langle \, \varphi(x_1) \, \varphi(x_2) \, \rangle^{(0)} + \langle \, \varphi(x_1) \, \varphi(x_2) \, \rangle_C^{(1)} \\
&\quad + \langle \, \varphi(x_1) \, \rangle^{(1)} \, \langle \, \varphi(x_2) \, \rangle^{(1)} + O(g^3) \, .
\end{aligned}$$

The connected part is given by

$$
\begin{aligned}
&\langle\, \varphi(x_1)\, \varphi(x_2)\, \rangle_C^{(1)} \\
&= \; - g^2 \int \frac{d^d p}{(2\pi)^d} \, \frac{e^{-ip\,(x_1 - x_2)}}{(\, p^2 + m^2\,)^2} \left[-\frac{1}{2} \int \frac{d^d \ell}{(2\pi)^d} \, \frac{1}{(\ell + p)^2 + m^2} \, \frac{1}{\ell^2 + m^2} \right. \\
&\qquad\qquad \left. + Z_\varphi^{(1)}\, p^2 + Z_m^{(1)}\, m^2 \right] \\
&\quad + g^2 \int \frac{d^d p}{(2\pi)^d} \, \frac{e^{-ip\,(x_1 - x_2)}}{m^2\,(\, p^2 + m^2\,)^2} \left[\frac{1}{2} \int \frac{d^d \ell}{(2\pi)^d} \, \frac{1}{\ell^2 + m^2} + Y^{(1)} \right].
\end{aligned}
$$



5. Schwinger-Dyson equations

Let us show that correlation functions described in terms of quantum A_∞ algebras satisfy the Schwinger-Dyson equations. Since

$$(\mathbf{I} + \boldsymbol{h}\, \boldsymbol{m} - \boldsymbol{h}\, \mathbf{U})\, \frac{1}{\mathbf{I} + \boldsymbol{h}\, \boldsymbol{m} - \boldsymbol{h}\, \mathbf{U}}\, \mathbf{1} = \mathbf{1}$$

and

$$\pi_{n+1}\, \mathbf{1} = 0 \quad \text{for} \quad n \geq 0\,,$$

we have

$$\pi_{n+1}\, \boldsymbol{f}\, \mathbf{1} + \pi_{n+1}\, \boldsymbol{h}\, \boldsymbol{m}\, \boldsymbol{f}\, \mathbf{1} - \pi_{n+1}\, \boldsymbol{h}\, \mathbf{U}\, \boldsymbol{f}\, \mathbf{1} = 0 \quad \text{for} \quad n \geq 0\,.$$

This equation is translated as

$$\begin{aligned}
&\langle\, \varphi(x_1) \ldots\, \varphi(x_n)\, \varphi(y)\, \rangle \\
&+ \sum_{k=0}^{\infty} \int d^d z \int \frac{d^d p}{(2\pi)^d}\, \frac{e^{-ip(y-z)}}{p^2+m^2}\, \langle\, \varphi(x_1) \ldots\, \varphi(x_n)\, m_k\, (\, \varphi(z) \otimes \ldots\, \otimes \varphi(z)\,)\, \rangle \\
&- \sum_{i=1}^{n} \int \frac{d^d p}{(2\pi)^d}\, \frac{e^{-ip\,(x_i-y)}}{p^2+m^2}\, \langle\, \varphi(x_1)\, \ldots\, \varphi(x_{i-1})\, \varphi(x_{i+1})\, \ldots\, \varphi(x_n)\, \rangle = 0\,.
\end{aligned}$$

We then acts the operator $-\partial_y^2 + m^2$ to find

$$
\begin{aligned}
&(-\partial_y^2 + m^2)\,\langle\, \varphi(x_1) \ldots \varphi(x_n)\, \varphi(y)\,\rangle \\
&+ \sum_{k=0}^{\infty} \langle\, \varphi(x_1) \ldots \varphi(x_n)\, m_k\,(\, \varphi(y) \otimes \ldots \otimes \varphi(y)\,)\,\rangle \\
&- \sum_{i=1}^{n} \delta^d(y - x_i)\,\langle\, \varphi(x_1)\, \ldots\, \varphi(x_{i-1})\, \varphi(x_{i+1})\, \ldots\, \varphi(x_n)\,\rangle = 0\,.
\end{aligned}
$$

Since

$$\frac{\delta S}{\delta \varphi(y)} = (\, - \partial_y^2 + m^2 \,) \, \varphi(y) + \sum_{k=0}^{\infty} m_k \, (\, \varphi(y) \otimes \ldots \otimes \varphi(y) \,) \, ,$$

we find

$$- \sum_{i=1}^{n} \delta^d (y - x_i) \, \langle \, \varphi(x_1) \, \ldots \, \varphi(x_{i-1}) \, \varphi(x_{i+1}) \, \ldots \, \varphi(x_n) \, \rangle$$
$$+ \langle \, \varphi(x_1) \, \ldots \, \varphi(x_n) \, \frac{\delta S}{\delta \varphi(y)} \, \rangle = 0 \, .$$

We have thus shown that the Schwinger-Dyson equations are satisfied.

6. Renormalization group

The construction of $\boldsymbol{h}$ from h is not unique. In addition to $\mathbf{P}$ for $P = 0$, let us introduce $\mathbf{P}_\Lambda$ for the projection onto modes below the energy scale Λ, and use $\boldsymbol{h}$ given by

$$\boldsymbol{h} = \boldsymbol{h}_H + \boldsymbol{h}_L \,,$$

where the propagator $\boldsymbol{h}_H$ for high-energy modes satisfy

$$\mathbf{Q}\,\boldsymbol{h}_H + \boldsymbol{h}_H\,\mathbf{Q} = \mathbf{I} - \mathbf{P}_\Lambda \,, \qquad \boldsymbol{h}_H\,\mathbf{P}_\Lambda = 0 \,, \qquad \mathbf{P}_\Lambda\,\boldsymbol{h}_H = 0 \,, \qquad \boldsymbol{h}_H^2 = 0$$

and the propagator $\boldsymbol{h}_L$ for low-energy modes satisfy

$$\mathbf{Q}\,\boldsymbol{h}_L + \boldsymbol{h}_L\,\mathbf{Q} = \mathbf{P}_\Lambda - \mathbf{P} \,, \quad \boldsymbol{h}_L\,(\,\mathbf{I} - \mathbf{P}_\Lambda\,) = 0 \,, \quad (\,\mathbf{I} - \mathbf{P}_\Lambda\,)\,\boldsymbol{h}_L = 0 \,, \quad \boldsymbol{h}_L^2 = 0 \,.$$

Then we can write $\boldsymbol{f}\,\mathbf{P}$ as

$$\begin{aligned}
&\frac{1}{\mathbf{I}+\boldsymbol{h}\,\boldsymbol{m}-\boldsymbol{h}\,\mathbf{U}}\,\mathbf{P} \\
&=\frac{1}{\mathbf{I}+\boldsymbol{h}_H\,\boldsymbol{m}-\boldsymbol{h}_H\,\mathbf{U}}\left(\mathbf{I}+\boldsymbol{h}_L\,(\,\boldsymbol{m}-\mathbf{U}\,)\,\frac{1}{\mathbf{I}+\boldsymbol{h}_H\,\boldsymbol{m}-\boldsymbol{h}_H\,\mathbf{U}}\right)^{-1}\mathbf{P} \\
&=\frac{1}{\mathbf{I}+\boldsymbol{h}_H\,\boldsymbol{m}-\boldsymbol{h}_H\,\mathbf{U}}\,\mathbf{P}_\Lambda\,\frac{1}{\mathbf{I}+\boldsymbol{h}_L\,\boldsymbol{m}_\Lambda-\boldsymbol{h}_L\,\mathbf{U}}\,\mathbf{P}\,,
\end{aligned}$$

where

$$\boldsymbol{m}_\Lambda=\mathbf{P}_\Lambda\left[\,(\,\boldsymbol{m}-\mathbf{U}\,)\,\frac{1}{\mathbf{I}+\boldsymbol{h}_H\,\boldsymbol{m}-\boldsymbol{h}_H\,\mathbf{U}}+\mathbf{U}\right]\mathbf{P}_\Lambda\,.$$

The operator $\boldsymbol{m}_\Lambda$ describes the Wilsonian effective action at the energy scale Λ, and correlation functions are calculated from a product of the operator for high-energy modes and the operator for low-energy modes.

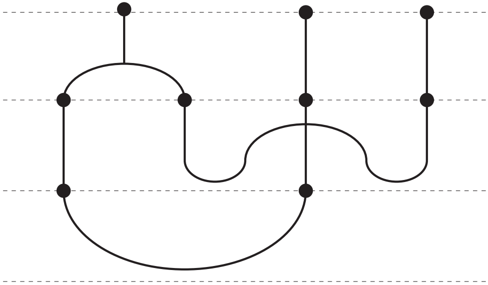
We can introduce a sequence of projections and write $\boldsymbol{f}\,\mathbf{P}$ as

$$\frac{1}{\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m} - \boldsymbol{h}\,\mathbf{U}}\,\mathbf{P} = \prod_i \frac{1}{\mathbf{I} + \boldsymbol{h}_i\,\boldsymbol{m}_i - \boldsymbol{h}_i\,\mathbf{U}}\,\mathbf{P}_i$$

with

$$\boldsymbol{h} = \sum_i \boldsymbol{h}_i\,.$$

While perturbative expressions for correlation functions with the previous $\boldsymbol{h}$ do not converge, this choice of $\boldsymbol{h}$ may lead to nonperturbative expressions for correlation functions.



7. Conclusions and discussion

We proposed the formula

$$\langle\, \varphi(x_1)\, \varphi(x_2)\ \ldots\ \varphi(x_n)\, \rangle = \omega_n\, (\, \pi_n\, \boldsymbol{f}\, \mathbf{1}\, , \delta^d(x-x_1)\otimes\delta^d(x-x_2)\otimes\ldots\otimes\delta^d(x-x_n)\,)$$

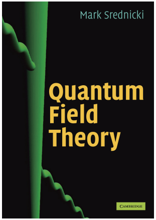
for correlation function of scalar field theories in perturbation theory using quantum A_∞ algebras.

We then proved that correlation functions from our formula satisfy the Schwinger-Dyson equations as an immediate consequence of the structure

$$(\, \mathbf{I} + \boldsymbol{h}\, \boldsymbol{m} - \boldsymbol{h}\, \mathbf{U}\,)\, \frac{1}{\mathbf{I} + \boldsymbol{h}\, \boldsymbol{m} - \boldsymbol{h}\, \mathbf{U}}\, \mathbf{1} = \mathbf{1}\, .$$

Since the description in terms of homotopy algebras or the Batalin-Vilkovisky formalism tends to be elusive and formal, we have presented completely explicit calculations for φ^3 theory which involve renormalization at one loop.

We hope that this demonstration in this paper helps us convince ourselves that any calculations of this kind in the path integral or in the operator formalism can be carried out in the framework of quantum A_∞ algebras as well.



While the expressions for correlation functions in terms of homotopy algebras are universal, our expressions are restricted to the case where $\mathcal{H}$ consists of only two sectors $\mathcal{H}_1$ and $\mathcal{H}_2$. It would be important to extend our analysis to more general cases.

We can extend the formula for correlation functions to incorporate Dirac fermions.

Konosu and Okawa, in progress

Our ultimate goal is to provide a framework to prove the AdS/CFT correspondence using open string field theory with source terms for gauge-invariant operators. The quantum treatment of open string field theory must be crucial for this program, and we hope that quantum A_∞ algebras will provide us with powerful tools in this endeavor.