On the Relationship Between Super-Riemann Surfaces and PCOs

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Peturbative String Theory

- Two formulations for RNS string theory amplitude, from different gauge fixings
  - Super-Riemann Surface (SRS)
  - Picture Changing Operator (PCO)
- PCO has subtleties
- Will show equivalence of the SRS and PCO formalisms
  - Based on 2205.01106, 2205.10377
- Setting: Type II superstring theory in flat space at weak coupling

For both formalisms, scattering amplitude is of the form

$$\mathcal{A}(\mathcal{V}_i) = \sum_{g,\epsilon,\bar{\epsilon}} \frac{g_s^{2g-2}}{2^{2g}} \int_{\mathcal{M}_{g,n,\epsilon,\bar{\epsilon}}} \Omega_{g,n,\epsilon,\bar{\epsilon}}(\mathcal{V}_i) \qquad (1)$$

The Worldsheet CFT

Recall the structure of the RNS type II superstring worldsheet theory:

- Super-CFT: Super-Virasoro algebra, fermionic stress tensor $G(z)$
- Matter: X^μ, ψ^μ, Ghosts: b, c, β, γ
- Nilpotent worldsheet BRST operator Q_B
  - $Q_B(b) = T, Q_B(\beta) = G$
- Also have $\delta(\beta)$ and $\delta(\gamma)$ operators
  - $\delta\left(\oint dz\, f(z)\beta(z)\right)$ is also well defined

Can rewrite ghosts in terms of rebosonized operators: ξ, η, ϕ

- $\partial\xi = -\partial\beta\,\delta(\beta)$, $\partial\phi = \beta\gamma$
- Have explicit formula for correlators

The SRS Formalism

Integrate over super-moduli space of SRS's, $\mathfrak{M}$. Integrand is

$$\Omega_{g,n,\epsilon} = \left\langle \prod_k \mathcal{B}_{t^k} dt^k \prod_a \delta(\mathcal{B}_{\nu^a})\delta(d\nu^a) \prod_{i=1}^{n} \mathcal{V}_i \right\rangle_{\mathcal{S}_{g,n,\epsilon}(t,\nu)} \tag{2}$$

Super-Riemann Surface (SRS): Gluing together patches with coordinates (z_i, θ_i) using transition maps

$$z_i = f_{ij}(z_j) + \theta_j g_{ij}(z_j) h_{ij}(z_j), \tag{3}$$

$$\theta_i = g_{ij}(z_j) + \theta_j h_{ij}(z_j), \tag{4}$$

where $h_{ij}^2 = \partial f_{ij} + g_{ij}\partial g_{ij}$.
Bosonic parameters: $g_{ij} = 0$ gives Riemann Surface with spin structure

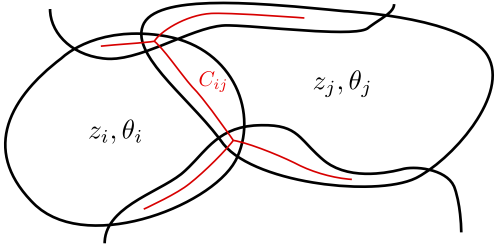
The SRS Formalism

$\mathcal{B}$ terms are contour integrals:

$$\sum \mathcal{B}_{t^k} dt^k + \sum \mathcal{B}_{\nu^k} d\nu^k = \tag{5}$$

$$\sum_{(ij)} \int_{C_{ij}} \frac{[dz_i | d\theta_i]}{2\pi i} (\beta(z) + \theta b(z)) \left[\delta z_i - \delta\theta_i \theta_i\right]\big|_{z_j, \theta_j} \tag{6}$$

Correlators for non-split SRS ($g_{ij} \neq 0$) can be calculated by inserting similar G contours.



The PCO Formalism

- Integrate over bosonic moduli space of Riemann Surfaces with spin structure, $\mathcal{M}_{g,n,\epsilon}$, also pick section of bundle $\pi : \mathcal{Y} \to \mathcal{M}$ of PCO locations.
- PCO:

$$\mathcal{X}(z) = Q_B(\xi(z)) \tag{7}$$

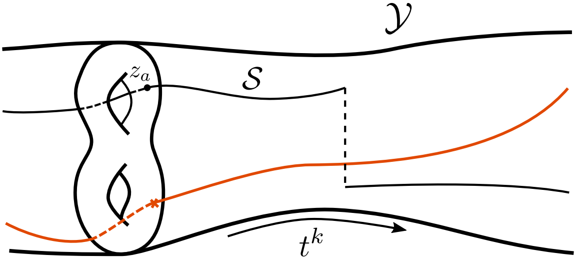
$$= \frac{1}{2}\oint \frac{dw}{2\pi i}(w-z)^{-1} G(w)\delta(\beta(z)) - \frac{1}{4}\partial\beta(z)\delta'(\beta(z)) \tag{8}$$

- Integrand is d_e-form over $\mathcal{Y}$,

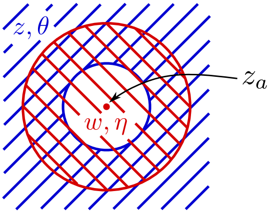
$$\widetilde{\Omega} = \left\langle e^{\pi^*\mathcal{B}} \prod_{a=1}^{d_o} [\mathcal{X}(z_a) + d\xi(z_a)][\widetilde{\mathcal{X}}(\bar{z}_a) + d\widetilde{\xi}(\bar{z}_a)] \prod_{i=1}^{n} \mathcal{V}_i \right\rangle_{\Sigma,\epsilon} \tag{9}$$

Vertical Integration

- Section $\mathcal{S}$ of $\mathcal{Y}$ must not intersect locus of spurious singularities
  - Can only be done locally in patches
- Integration contour must be closed using 'vertical segments' with $\xi(z_1) - \xi(z_2)$ terms.



Codimension 0 – Disk Cutting



- Given PCO location z_a
- Introduce new coordinates parameterizing the neighborhood around z_a, (w, η), with transition map

$$w = z - \frac{\theta \nu^a}{z - z_a}, \tag{10}$$

$$\eta = \theta - \frac{\nu^a}{z - z_a} \tag{11}$$

Codimension 0 – Disk Cutting

SRS integrand:

$$\left[1 + \nu \oint \frac{dz}{2\pi i} \frac{G(z)}{z - z_a}\right] \left[1 - 2\nu\partial\beta(z_a) dz_a\right] \cdot \delta(d\nu) \left[\frac{1}{2}\delta(\beta(z_a)) - \frac{1}{4}\nu\partial b(z_a)\, \delta'(\beta(z_a))\right] \tag{12}$$

- Integration over ν gives $\mathcal{X}(z_a) + d\xi(z_a)$

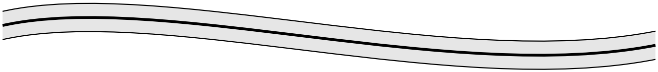
Supermanifolds

- Grassmann odd directions in supermanifolds not quite 'real' in sense that there are not different points corresponding to different values of coordinates
  - Supermanifolds are purely abstract gadget for purpose of integration
- Fermionic directions are infinitesimal. All points have all fermionic coordinates zero
  - Consider a bosonic coordinate ϵ satisfying $\epsilon^2 = 0$ – Just calculates first order in power expansions, so is infinitesimal
  - $(\nu_1 \nu_2)^2 = 0$ for anticommuting ν_1, ν_2
- Algebraic Geometry point of view: geometry defined from ring of functions on local patch
- Functions on supermanifold defined as a formal power series

$$f(x^i, \theta^a) = f_0(x^i) + \theta^a f_{1,a}(x^i) + \theta^a \theta^b f_{1,ab}(x^i) + \dots \quad (13)$$

Supermanifolds

The heuristic picture is:
A supermanifold is a normal manifold thickened by infinitely thin fermionic 'fuzz'

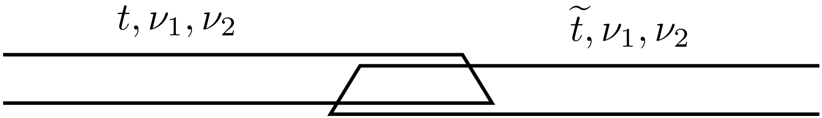


Generically not a fiber bundle!

Supermanifolds

- Manifold constructed by gluing together patches of $\mathbb{R}^n$.
- Supermanifold $\mathfrak{M}$ constructed by gluing together patches of $\mathbb{R}^{n|m}$
  - Super-manifold with bosonic transition maps has setting all $\nu_a = 0$ independent of patch.
  - Bosonic manifold $\mathcal{M}^{\mathrm{red}}$ from forgetting fermionic coordinates, have inclusion $\mathcal{M}^{\mathrm{red}} \hookrightarrow \mathfrak{M}$
- Example: glue together patches $(t, \nu_1, \nu_2), t < \delta$ and $(\widetilde{t}, \nu_1, \nu_2), t > -\delta$ with transition map

$$\widetilde{t} = t + \nu_1 \nu_2 \tag{14}$$



Integration in Supermanifolds

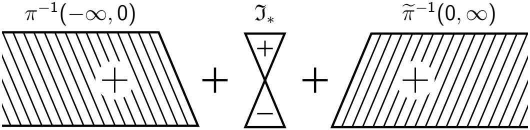
- Integral form is defined on patches, must match with 'Jacobian' (Berezinian) rescaling on overlap
- Example:

$$\omega = [dt|d\nu_1 d\nu_2](f(t) + g(t)\nu_1\nu_2) \tag{15}$$
$$= [d\tilde{t}|d\nu_1 d\nu_2](f(\tilde{t}) + (g(\tilde{t}) - f'(\tilde{t}))\nu_1\nu_2) \tag{16}$$

- Integrate using partition of unity: $Supp(f_a) \subset \mathfrak{U}_a$, $\sum_a f_a = 1$,

$$\int_{\mathfrak{M}} \omega = \sum_a \int_{\mathfrak{U}_a} f_a \, \omega \tag{17}$$

Interpolation



- Two pieces – (t, ν_1, ν_2), $t < \delta$; $(\tilde{t}, \nu_1, \nu_2)$, $\tilde{t} > -\delta$, transition map is

$$\tilde{t} = t + \nu_1 \nu_2 \tag{18}$$

- Integral form:

$$\omega = [dt|d\nu_1 d\nu_2](f(t) + g(t)\nu_1\nu_2) \tag{19}$$

$$= [d\tilde{t}|d\nu_1 d\nu_2](f(\tilde{t}) + (g(\tilde{t}) - f'(\tilde{t}))\nu_1\nu_2) \tag{20}$$

- Just need to cancel boundary: interpolation piece: $s \in [0, 1]$, $t = -s\nu_1\nu_2$.

$$\mathfrak{I}^*\omega = -[ds|d\nu_1 d\nu_2]\nu_1\nu_2 f(0) \tag{21}$$

Interpolation

- Integral:

$$\int_{-\infty}^{0} g(t)\ dt + \int_{0}^{1} f(0)\ ds + \int_{0}^{\infty} (g(\tilde{t}) - f'(\tilde{t}))\ d\tilde{t} \tag{22}$$

$$= \int_{-\infty}^{\infty} g(t)\ dt \ - \ f(+\infty) \tag{23}$$

- Interpolation term needed for consistent result

Interpolation

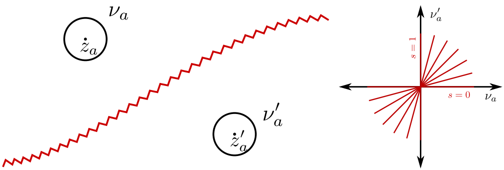
More generally:

- Split reduced space $\mathcal{M}$ based on dual triangulation into domains $\mathcal{D}_i$, pick coordinates on each patch.
- For each patch, integrate as usual Berezinian integral – integrating over fermionic fibers of constant bosonic coordinates, then over the bosonic moduli
- To close integration contour, must also integrate pullback of the integral form along

$$\mathfrak{I}_{i_1 \cdots i_{p+1}} : \Delta^p \times \mathcal{D}_{i_1 \cdots i_{p+1}} \times \mathbb{R}^{0|n_o} \to \mathfrak{M} \tag{24}$$

over the interpolating piece at the codimension p interface between $p+1$ domains.

Codimension 1 – Moving a Single PCO



- Two disks centered at z_a and z'_a, with parameters ν_a and ν'_a
- Interpolation coordinates s, ν''_a,

$$\nu_a = R(1-s)\nu''_a \tag{25}$$

$$\nu'_a = R's\nu''_a \tag{26}$$

- R and R' chosen to avoid coordinate degenerations. Rescaling does not affect fiber on boundary.
- Integrating over s and ν''_a returns $\xi(z'_a) - \xi(z_a)$
- If multiple PCO's need to move, move one at a time.

Codimension ≥ 2

- Similar approach for higher codimension – split interpolating contour into discrete moves, match moves with vertical integration
- E.g., in codimension 2, have triangle moves with three locations of a single PCO (evaluates to 0), square moves commuting moves of two different PCO's (evaluates to product of $\xi(z_2) - \xi(z_1)$)

Summary

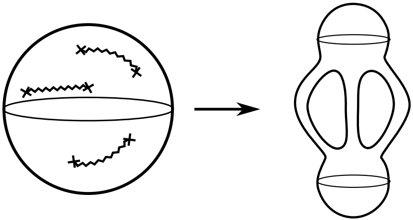
The correspondence between the SRS and PCO formalisms:

- PCO integrand can be interpreted as an integral over specific choice of fermionic fibers for SRS super-moduli space
- Mismatch of fibers between patches requires interpolating segments
- Vertical integration terms equal contribution from interpolating segments

Application: Genus Two

Explicit construction of PCO coordinates for $\mathfrak{M}$ in genus two case:

- Convenient parameterization of bosonic moduli space using double covers with six branch points
- Spin structure is partition of branch points into two types. $(3, 3)$ split is even spin structre, $(5, 1)$ split is odd.
- Branched cover can be constructed by equation $y^2 = f(x)$, f is a degree 6 polynomial.



SRS at Genus Two

- Alternate description of SRS structure: 'totally non-integrable rank $0|1$ sub-bundle of the tangent bundle'
  - Fermionic differential operator D, defined up to local rescaling, $\{D, D\}$ is nowhere zero
- Spin structure is factorization $f(x) = p(x)q(x)$. Let $\alpha = y/p(x) = q(x)/y$.
- Split SRS coordinates:
  - η, non-degenerate where $p(x) \neq 0$
  - τ, non-degenerate where $q(x) \neq 0$
- Transition map is $\tau = \alpha\eta$
- $D_\eta = \partial_\eta + \alpha\eta\partial_x = \alpha D_\tau$

Disc Cutting

- Place both PCO's on majority type branch points for odd spin structure, one on each type for even spin structure
- On a $q = 0$ branch point, disc coordinates are

$$w_a = y - \frac{f'(x)}{2p(x)} \eta \frac{f'(x)\nu_a}{y} \tag{27}$$

$$\eta_a = \eta - \frac{f'(x)\nu_a}{y} \tag{28}$$

Odd Spin Structure

- Starting PCO locations: x_1 and x_2
- Final PCO locations: x_3 and x_2

Transition map is:

$$(x_i', \nu_1', \nu_2') = \left(x_i, \frac{x_{12}}{x_{32}} \nu_1, \nu_2 + \frac{x_{13}}{x_{23}} \right) \tag{29}$$

- No change in fibers when changing PCO locations!
- Vertical integration contribution vanishes

Even Spin Structure

- Branch point x_2 moves when the moving PCO at x_1:

$$(x_2', \nu_1', \nu_2') = \left(x_2 - 2f'(x_2) \frac{x_{31}}{x_{23} x_{21}} \nu_1 \nu_2, \nu_1, \nu_2 \right) \qquad (30)$$

- Non-zero vertical integration contribution
- Can also match with Period Matrix projection used by D'Hoker and Phong